Tic-Tac-Toe or Is the dual of an algebraic matroid algebraic?

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1 Introduction

We present a matroid that has some interesting properties related to the theory of algebraic matroids.

Algebraic matroids, although already known from the very beginning of matroid theory [7] are far less understood than matroids that are linearly representable over some field. This is illustrated by the fact that it is still not known whether the dual of an algebraic matroid is algebraic or not, although this problem has been posed in several prominent places.

Definition 1. *Let k be a field and $k \subseteq K$ a field extension. A finite subset $\{v_1, \dots, v_k\} \subseteq K$ is called* algebraically dependent *over k if there exists a non-trivial polynomial $0 \neq p(x_1, \dots, x_k) \in k[x_1, \dots, x_k]$ such that*

$$p(v_1, \dots, v_k) = 0.$$

The set K is called algebraically independent*, otherwise.*

Algebraic independency satisfies the axioms of matroid theory (see e.g. 6.7.1 in [6]).

Definition 2. *A matroid $M = (E, \mathcal{I})$ on a finite set E is called* algebraic*, if there exists a field extension $k \subseteq K$ a subset $E' \subseteq K$ and a bijection $\sigma : E \to E'$ such that for all $I \subseteq E$*

$$I \in \mathcal{I} \Leftrightarrow \sigma(I) \text{ is algebraically independent over } k.$$

Tt has been shown by Pfiff (1969) that algebraic matroids are a superclass of the class of linear matroids and by Ingleton (1971) that this superclass is proper (see [6] 6.7.10). It took until 1975 to show that the Vámos matroid is an example of a non-algebraic matroid [4].

2 Combinatorial Properties of Algebraic Matroids

The key lemma in the proof that the Vámos matroid is non-algebraic was generalized by A. Dress and L. Lovász to the "series reduction theorem" [3].

Definition 3. *Let $M = (E, \mathcal{I})$ be a matroid on a finite set E and $S \subseteq A \subseteq E$. Then S is* in series in A *if contracting $A \setminus S$ turns S into a circuit.*

Theorem 4 (Dress-Lovász 1987). *Let $M = (E, \mathcal{I})$ be an algebraic matroid represented by a set $E' \subseteq K$ over a field k, S' in series in $A' \subseteq E'$ and K algebraically closed. Then there exists $\beta \in K$ such that $\forall T' \subseteq A' \setminus S'$:*

$$S' \cup T' \text{ is algebraically dependent}$$
$$\Updownarrow$$
$$\beta \cup T' \text{ is algebraically dependent.}$$

From this it is seen that the Vámos matroid is non-algebraic as follows:

The Vámos matroid V is defined on an eight point set $\{a, a', b, b', c, c', d, d'\}$ by its bases. All four point sets are bases except for the five four point planes $\{\{a, a', b, b'\}, \{a, a', c, c'\}, \{b, b', c, c'\}, \{b, b', d, d\}$, $\{c, c', d, d'\}\}$. In particular $\{a, a', d, d'\}$ is independent. Assume V were algebraically represented over a field extension $k \subseteq K$. Since $\{a, a'\}$ is in series in $\{a, a', b, b', c, c'\}$ there exists a β_1 in the algebraic closure of K which lies on the intersection of the two "lines" $\beta_1 \in \overline{b, b'} \cap \overline{c, c'}$. We denote the closure of $V \cup \beta_1$ by overlining the sets. We also have $\beta_1 \in \overline{a, a', b, b'} \cap \overline{a, a', c, c'} = \overline{a, a'}$. By symmetry there also exists $\beta_2 \in \overline{a, a'} \cap \overline{b, b'} \cap \overline{d, d'}$. Since β_1 and β_2 lie in the intersection of the same lines they must be parallels. Thus the two lines $\overline{a, a'}$ and $\overline{d, d'}$ have an intersection of rank at least one, contradicting the independence of the set $\{a, a', d, d'\}$.

All proofs of non-algebraicity, known to the author, apply the series reduction theorem, postulate additional points and derive a contradiction.

3 The Tic-Tac-Toe Matroid

A matroid that already has all the points required by Theorem 4 is called *pseudomodular* [2]. The Tic-Tac-Toe matroid presented below is pseudomodular and its dual is non-algebraic.

The Tic-Tac-Toe matroid (see Figure 3) is defined as a rank five matroid on the points $\{1,2,3,4,5,6,7,8,9\}$. Any five points are independent except for the eight "five point hyperplanes" $\{1,2,3,4,7\}$, $\{1,2,3,5,8\}$, $\{1,2,3,6,9\}, \{4,5,6,1,7\}, \{4,5,6,3,9\}, \{7,8,9,1,4\}, \{7,8,9,2,5\}, \{7,8,9,3,6\}$. These are the sets that have the shape of an L or a T in Figure 3. Note, that $\{4,5,6,2,8\}$ is independent.

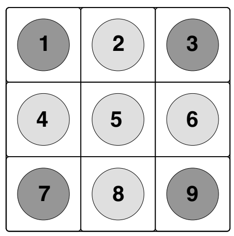

Figure 3: The Tic-Tac-Toe Matroid

It is proven in [1] that this matroid is pseudomodular. We present a proof that its dual is non-algebraic following [5]. This fact had been observed before, independently by M. Alfter and by B. Lindström. Note, that the dual consists of "three prisms in a triangle with one broken hyperplane $\{1,3,7,9\}$". In particular $\{5,8,6,9\}$, $\{4,7,6,9\}$ and $\{4,7,5,8\}$ form such a prism as well as $\{2,5,3,6\}$, $\{1,4,3,6\}$ and $\{1,4,2,5\}$. Assume this matroid were algebraic. According to the series reduction theorem there exist $a \in \overline{4,7} \cap \overline{5,8} \cap \overline{6,9}$ and $a' \in \overline{1,4} \cap \overline{2,5} \cap \overline{3,6}$. Since $\{a,a'\}$ is a subset of $\overline{1,4,7}$ as well as of $\overline{2,5,8}$ and $\overline{3,6,9}$, the eight points $\{1,2,3,7,8,9,a,a'\}$ form a Vámos matroid, contradicting algebraic representability.

4 Conclusion

Is the Tic-Tac-Toe matroid algebraic? Known techniques do not suffice to prove it is not. If it where algebraic, this would settle the old question whether the dual of an algebraic matroid is algebraic to the negative. Up to now, we did not have any better idea, than to search for algebraic coordinates with trial and error. Unfortunately we did not get beyond error.

References

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