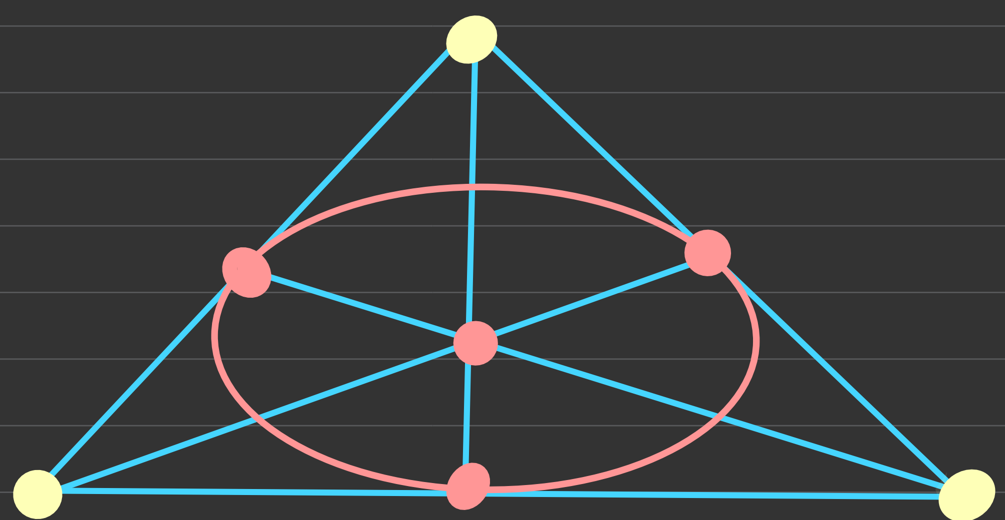
Unavoidable flats in matroids with large rank

Jim Geelen

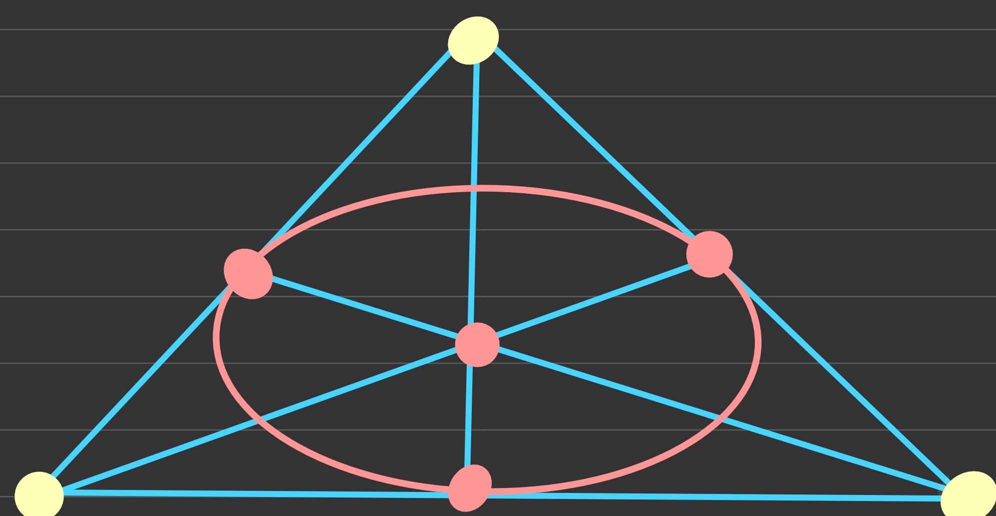
University of Waterloo

In any 2-colouring of the points of F_7 there is a monochromatic line.



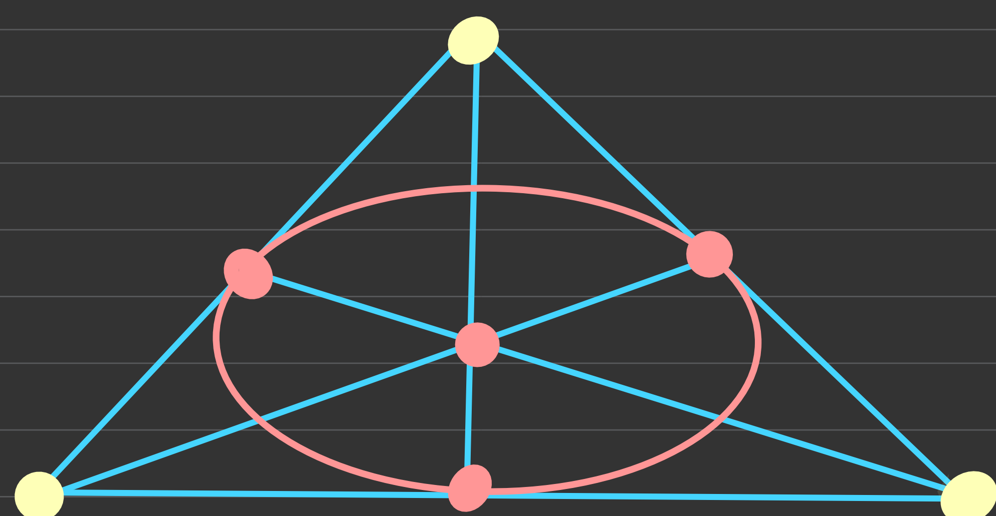
Conjecture (Ramsey's Theorem for matroid lines).

For $r \gg \ell$, if we 2-colour the elements of a simple rank-r matroid M with no lines of length $> \ell$, then there is a monochromatic line.



Conjecture (Ramsey's Theorem for matroids):

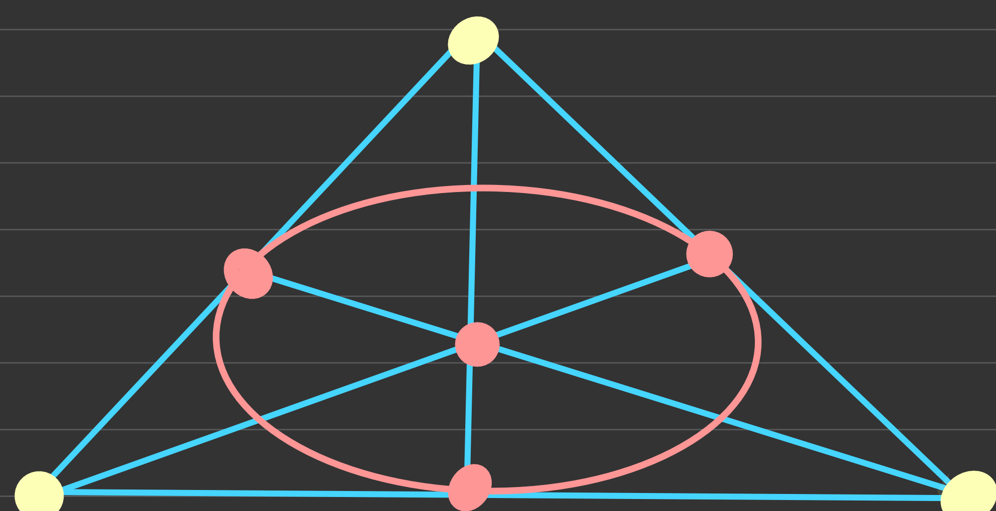
For $r \gg \ell + k$, if we 2-colour the points of a rank-k matroid with no lines of length $> \ell$, then M has a monochromatic rank-k flat.



Conjecture (Ramsey's Theorem for matroids):

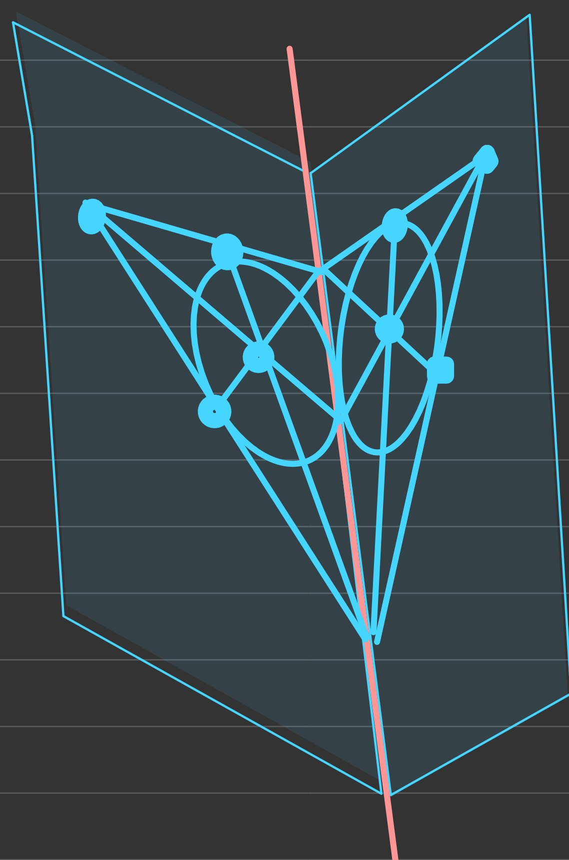
For $r \gg l+k$, if we 2-colour the points of a rank-r matroid with no lines of length $> l$, then M has a monochromatic rank-k flat.

Why lines, not flats of higher rank?



FALSE

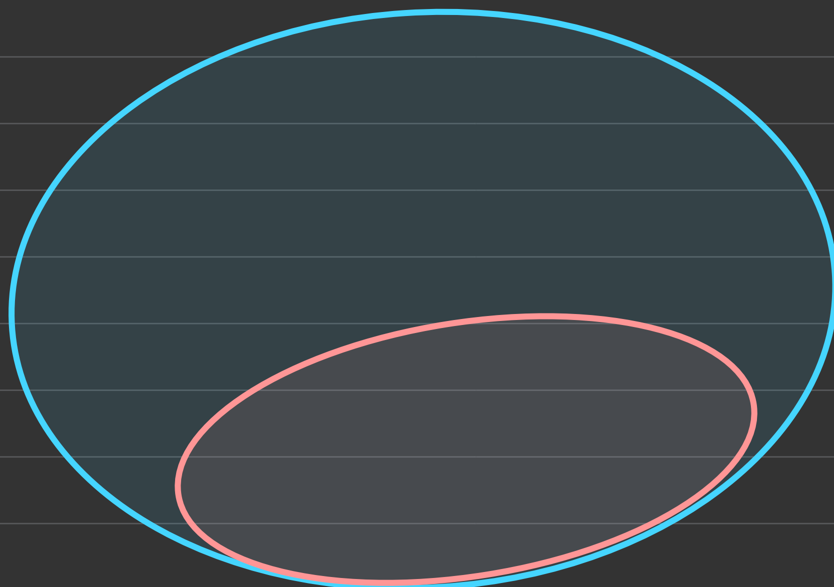
~~Conjecture:~~ If M is a simple rank-4 matroid with no line of length ≥ 3, then M contains a plane of size ≤ 4.



$AG(3, \mathbb{F}_2)$

FALSE

~~Conjecture~~: If M is a simple rank-4 matroid with no line of length $\geq q+1$, then M contains a plane of size $\leq q^2$.



$AG(3, \mathbb{F}_q)$

FALSE

~~Main Conjecture:~~ If M is a simple matroid with no line of length $\geq q+1$, then there is a hyperplane H such that for each flat F in H

$$|F| \leq q^{r(F)}.$$

Counterexamples (Bonin and Van der Pol)

For all k there exist infinitely many integers n such that there is a simple, n-element, triangle free matroid with rank 4 such that every plane contains exactly k points.

Proof: Such matroids correspond naturally with Steiner systems $S(3,k,n)$ which are known to exist for infinitely many n thanks to results of Keevash. $\square$

Now what?

New Conjecture: If M is a simple triangle-free matroid with sufficiently large rank, then M has a plane with at most 4 points.