Dependencies Among Dependencies

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Derived Matroids

Rota '71: Investigate "dependencies among dependencies" in matroids.

$$M = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|cccc} & \{a,b\} & \{a,c\} & \{b,c\} & \{d\} \\ a & 1 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

Definition (Longyear '80)

Given a binary matroid M, the *derived matroid* δM of M is the binary matroid on the set of circuits of M where a set $\mathcal{C}$ of circuits of M is independent in δM if and only if every non-empty subset of $\mathcal{C}$ has non-empty symmetric difference.

Open Problems

1. Which binary matroids are derived?
   - $U_{n,n+1}$ is not derived when $n \geq 3$.
   - $\mathrm{PG}(n-1, 2)$ is derived when $n \geq 1$.

2. Which binary matroids are derived from a graphic matroid? (or cographic, regular, etc.)

Structural Properties of Derived Matroids

Proposition (Oxley, Walsh)

If M is connected, then δM has an $M(K_{r(\delta M)+1})$-restriction.

Open Problems:

1. If M is connected, is each pair of elements of δM in a common circuit of size at most four?
2. Equivalently, if a binary matroid M has a partition into a pair of spanning circuits, does it have at least two such partitions?

Thomason '78: 'Yes' when M is a graphic.