Minimum Linear Arrangement (MLA)

- Input: a graph $G = (V, E)$ on n vertices.
- Output: an ordering of V, i.e. $\sigma \in \mathcal{S}_V$ where $\mathcal{S}_V$ is the set of bijective functions from $V(G)$ to $\{1, \ldots, n\}$.
- Objective: minimize $\sum_{uv \in E} |\sigma(u) - \sigma(v)|$.
- Intuition: Find an arrangement that minimizes total edge stretch.
- First introduced for designing error correcting codes. (Harper 1964)
- Proved to be NP-complete with a reduction involving the max cut problem. (Garey et al. 1974)
- Many alternative names: optimal linear ordering, edge sum, minimum 1-sum, bandwidth sum, or wirelength problem.

Minimum Sum Vertex Cover (MSVC)

- Input: a graph $G = (V, E)$ on n vertices.
- Output: an ordering of V, i.e. $\sigma \in \mathcal{S}_V$ where $\mathcal{S}_V$ is the set of bijective functions from $V(G)$ to $\{1, \ldots, n\}$.
- Objective: minimize $\sum_{uv \in E} \min\{\sigma(u), \sigma(v)\}$.
- Intuition: Find a vertex cover that covers $E(G)$ the most "efficient."
- Introduced as a heuristic in solving semidefinite relaxation of the max cut problem. (Burer, Monteiro 2001)
- Known to NP-hard to approximate for some constant greater than 1. (Feige,Lovász,and Tetali 2004)

Minimum Latency Vertex Cover (MLVC)

- Input: a graph $G = (V, E)$ on n vertices.
- Output: an ordering of V, i.e. $\sigma \in \mathcal{S}_V$ where $\mathcal{S}_V$ is the set of bijective functions from $V(G)$ to $\{1, \ldots, n\}$.
- Objective: minimize $\sum_{uv \in E} \max\{\sigma(u), \sigma(v)\}$.
- Intuition: Consider the vertices and edges as jobs with precedence constraints, find the most efficient job scheduling.
- A special case of the minimum latency set cover problem. (Hassin,Levin 2005)
- MLVC is NP-hard. (Farhadi, Gupta, Sun, Tetali, Wigal 2021)

The Problem

MLA: $\min_\sigma \sum_{uv\in E} |\sigma(u) - \sigma(v)|$ **MSVC**: $\min_\sigma \sum_{uv\in E} \min\{\sigma(u), \sigma(v)\}$ **MLVC**: $\min_\sigma \sum_{uv\in E} \max\{\sigma(u), \sigma(v)\}$

Theorem (Farhadi, Gupta, Sun, Tetali, Wigal 2021)

Let G be a graph on n vertices and $\sigma \in \mathcal{S}_V$.

$$\sum_{uv\in E(G)} \max\{\sigma(u), \sigma(v)\} = (n^3 - n)/3 - (n+1)|E(\overline{G})| + \sum_{uv\in E(\overline{G})} \min\{\sigma'(u), \sigma'(v)\}$$

where $\sigma' := n + 1 - \sigma$. If G is k-regular, then

$$k\binom{n+1}{2} + \sum_{uv\in E(G)} |\sigma(u) - \sigma(v)| = 2 \cdot \sum_{uv\in E(G)} \max\{\sigma(u), \sigma(v)\}.$$

Corollary (Farhadi, Gupta, Sun, Tetali, Wigal 2021)

For the family of regular graphs, MLA, MLVC, and MSVC are equivalent in decision form.

Problem

Are MLA, MSVC, and MLVC NP-hard for the family of regular graphs?