Pseudospherical Crossing Number

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The Harary-Hill Conjecture (1963) asserts that the crossing number $\operatorname{cr}(K_{2n+1})$ of the complete graph K_{2n+1} satisfies

$$\operatorname{cr}(K_{2n+1}) = H(2n+1) := \binom{n}{2}^2 .$$

- If this is true for K_{2n+1}, then an easy count shows that the corresponding conjecture

  $$\operatorname{cr}(K_{2n+2}) = H(2n+2) := \binom{n+1}{2}\binom{n}{2}$$

  for the next case is also true. We don't know how to go *even* $\Rightarrow$ *odd*.

- There are quite natural (and some less natural!) drawings that satisfy this conjecture.

Known:

- (Ábrego and Fernández-Marchant, 2005; Lovász, Vestergombi, Wagner, and Welzel, 2004) If D is a rectilinear=straightline (pseudolinear) drawing of K_n, then $\mathrm{cr}(D) \geq H(n)$ and, for $n = 8$ or $n \geq 10$, $\mathrm{cr}(D) > H(n)$.

- If D is a 2-page drawing of K_n, then $\mathrm{cr}(D) \geq H(n)$ and $H(n)$ is achieved by a 2-page drawing of K_n.

There is a direct line from the geometric argument for rectilinear to the combinatorial/topological version used for the 2-page result.

A drawing D of K_n in the sphere is *pseudospherical* if:

- for each edge e of K_n, there is a simple closed curve γ_e containing γ_e and disjoint from all the other vertices of K_n;

- for distinct edges e, f of K_n, $|\gamma_e \cap \gamma_f| \leq 2$ and each intersection either a vertex or a crossing;

- for distinct edges e, f, $|e \cap \gamma_f| \leq 1$

This is modelled on the structure induced when all the γ_e are great circles in the sphere (that is, a *spherical* drawing). In this case, $|\gamma_e \cap \gamma_f|$ equals 2.

Characterizations of Pseudospherical

The following are equivalent for a drawing D of K_n in the sphere (Arroyo, R., Sunohara, 2021):

- D is pseudospherical;

- D is pseudospherical and, for distinct edges e, f,

$$|\gamma_e \cap \gamma_f| = 2; \quad \text{and}$$

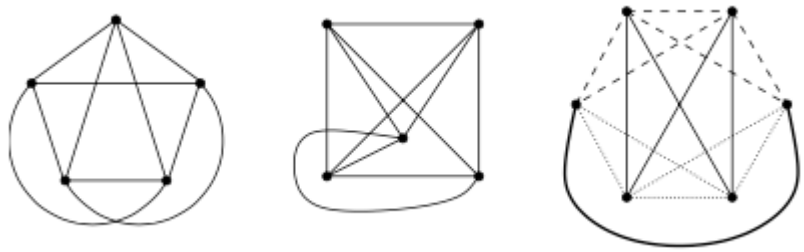
- D does not contain any of the illustrated subdrawings;

- D is hereditarily convex*.

REMARK: Most known optimal drawings are known to be pseudospherical and none has been shown to be not pseudospherical.

*A drawing of K_n in the sphere is *convex* if, for each 3-cycle T of K_n, there is a closed side Δ_T of T such that, if u, v are vertices on that side of T, then uv is also on that side.

A convex drawing is *hereditarily convex* if there are choices for all the Δ_T showing convexity and such that, if $T' \subseteq \Delta_T$, then $\Delta_{T'} \subseteq \Delta_T$.



Question:

Does every pseudospherical drawing of K_n have at least $H(n)$ crossings?