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If G *has no cycles of length* $1 \bmod k$*, then* G *is* k*-colourable.*

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If G *has no cycles of length* $1 \bmod k$, *then* G *is* k*-colourable.*

Theorem (M., West; Referee)

If G *has less than* $\frac{k!}{2}$ *cycles of length* $1 \bmod k$, *then* G *is* k*-colourable. Further, if* G *contains an edge* e *such that* $G - e$ *is* k*-colourable but* G *is not, then* e *lies in at least* $(k-1)!$ *cycles of length* $1 \bmod k$, *and* $G - e$ *contains at least* $\frac{(k-1)!}{2}$ *cycles of length* $0 \bmod k$.

Colourings = Orientations

Theorem (Minty)

A graph G is k-colourable if and only if there exists an orientation of G in which no cycle of G has more than $(k-1)$ times as many forward edges as backward edges.

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Theorem (Tuza)

A graph G is k-colourable if and only if there exists an orientation of G in which no cycle of length $1 \bmod k$ of G has more than $(k-1)$ times as many forward edges as backwards edges.

The question

Conjecture

A graph is k-colourable if and only if there exists an orientation of G such that at most $(k-1)!-1$ cycles of length $1 \bmod k$ have more than $(k-1)$ times as many forward edges as backwards edges.