Hadwiger-Nelson Problem (1950)

How many colours are needed to colour the plane so that no two points at unit distance are the same colour?

Theorem (1950 Isbell)

At most 7 colours are needed.

Theorem (2018 de Gray)

At least 5 colours are needed.

Ringel's Circle Problem (1959)

Let $\mathcal{C}$ be a finite collection of circles in the plane, no three tangent at a point. Let $G(\mathcal{C})$ be the graph with vertex set $\mathcal{C}$ and edge set being the pairs of tangent circles. Is there an upper bound on the chromatic number of G, and if so then what is the best possible upper bound? Is it 5?

If all circles have radius $1/2$, then these graphs are exactly unit distance graphs.

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Theorem (2021 Davies, Keller, Kleist, Smorodinsky, Walczak)

There is no finite upper bound for the chromatic number of G.

Given a subset of D of $\mathbb{R}^+$, let $\mathbb{R}^2(D)$ be the graph with vertex set $\mathbb{R}^2$ and edge set $\{ab : d(a,b) \in D\}$. So the unit distance graph is $\mathbb{R}^2(\{1\})$.

Rosenfeld's Odd-Distance Problem (1994)

Does $\mathbb{R}^2(\{1,3,5,7,9,\ldots\})$ have bounded chromatic number?

Distance graphs

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Gallai's Theorem (1933)

For every finite set $T \subset \mathbb{R}^n$, there exists a finite set $X \subset \mathbb{R}^n$ such that every k-colouring of X contains a monochromatic set T' obtainable from T by a scaling and a translation.