Symmetry-forced rigidity in the plane

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Website:
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Relevant preprint:

See also my guest post on the matroid union blog:
http://matroidunion.org/?p=3675
Rigidity theory basics

Definition

A bar and joint framework in \( d \) dimensions consists of

- a graph \( G \), and
- a function \( p : V(G) \rightarrow \mathbb{R}^d \).

Such frameworks can be rigid or flexible.

Example

Let \( G \) be the graph on vertex set \( V = \{1, 2, 3, 4\} \) with edges \( \{12, 23, 34, 14\} \). Below we give three functions \( p : V \rightarrow \mathbb{R}^2 \). The first two frameworks are flexible and the third one is rigid.
Generic rigidity

Let $G$ be the graph on vertex set $\{1, 2, 3, 4\}$ with all edges, aside from $\{1, 3\}$. For generic $p : \{1, 2, 3, 4\} \to \mathbb{R}^2$, the resulting framework is rigid.

**Definition**

A graph $G$ is *generically rigid in* $\mathbb{R}^d$ if for every generic $p : V \to \mathbb{R}^d$, the resulting framework is rigid. Such a graph is *minimal* if removing any edge destroys this property.

We will soon see that generically rigid graphs in $\mathbb{R}^d$ are the spanning sets of a certain matroid.
Classical results

Question
Which graphs are (minimally) generically rigid in $\mathbb{R}^d$?

Proposition (Folklore)
A graph is generically rigid in $\mathbb{R}^1$ if and only if it is connected.

Theorem (Pollaczek-Geiringer 1927, “Laman’s Theorem")
A graph $G$ is minimally generically rigid in $\mathbb{R}^2$ if and only if
1. $|E(G)| = 2|V(G)| - 3$, and
2. $|E(G')| \leq 2|V(G')| - 3$ for all subgraphs $G'$ of $G$.

Generic rigidity in 3 dimensions remains an open problem.
Symmetry-forced rigidity

Frameworks appearing in many applications have forced symmetry. Symmetry-forced rigidity ignores flexes that break the symmetry.

Main problem: Characterize generic symmetry-forced rigidity in $\mathbb{R}^d$. 
Gain graphs

Definition
Given a group $S$, a graph $G$ has $S$-symmetry if there exists a free action of $S$ on $V(G)$ such that the action of each element of $S$ is a graph isomorphism of $G$.

Symmetric frameworks can be compactly represented with gain graphs.

Definition
Given a group $S$, an $S$-gain graph is a directed multigraph $G$ whose arcs are labeled by elements of $S$.

$S = \{I, A, A^2, A^3\}$

$A$ is a rotation $90^\circ$ counterclockwise.
Main problem: For each $d$ and each subgroup $S$ of Euclidean isometries of $\mathbb{R}^d$, characterize the $S$-gain graphs that are minimally generically rigid

- Mathematical foundations: Schulze and Whiteley 2010, Borcea and Streinu 2010
- Combinatorial characterizations:
  - Two-dimensional (flexible) lattices – Malestein and Theran 2013; Ross 2015
  - Three-dimensional lattices for body-bar frameworks – Ross 2015
  - All rotation groups and odd dihedral groups – Malestein and Theran 2014; Jordán, Kaszanitzky, and Tanigawa 2016
  - Wallpaper groups with flexible lattices – Malestein and Theran 2015

This talk: tropical geometry, submodular functions, and matroid lifts to unify and generalize many aspects of the characterizations above
**Balanced cycles**

**Definition**

The *gain* of a walk $W$ in a gain graph $G$ is the product of the labels in $W$, inverting when an arc is traversed backwards. A *balanced cycle* is a cycle whose gain is the identity.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S = \{ I, A, A^2, A^3, R, RA, RA^2, RA^3 \}$$

Balancedness of a cycle not affected by:

- reversing an edge and inverting its label
- starting at a different vertex
Dutch bicycles and complete gain graphs

Definition

A *bicyclic graph* is a subdivision of one of the following graphs:

\[ \infty \quad \infty \quad \infty \]

A bicyclic gain graph is *Dutch* if each pair of closed walks based at the same vertex have gains that commute.

Definition

Given a group \( S \), the *complete gain graph* \( K_n(S) \) has vertex set \( \{1, \ldots, n\} \) and \( |S| \) arcs from \( i \) to \( j \) when \( i < j \) and \( |S| - 1 \) loops at each vertex. Each non-loop edge between \( i \) and \( j \) is labeled by a distinct element of \( S \) and each loop edge is labeled by a distinct non-identity element of \( S \).

\[ K_2(\mathbb{Z}_2) = \]

\[ 1 \quad 0 \quad 1 \]

\[ 1 \quad 0 \quad 1 \]
The main theorem

Theorem (DIB 2021)

Let \( S \) be a subgroup of \( \mathbb{R}^2 \rtimes SO(2) \). For each \( S \)-gain graph \( H \), define

\[
\alpha(H) = \begin{cases} 
3 & \text{if every cycle in } H \text{ is balanced} \\
2 & \text{if not, and the gain of each cycle is a translation} \\
1 & \text{if none of the above, and all bicyclic subgraphs are Dutch} \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( G \) is minimally generically infinitesimally rigid in \( \mathbb{R}^2 \) if and only if

\[
|E(G)| = 2|V(G)| - \alpha(K_{|V(G)|}(S))
\]

and for all subgraphs \( G' \) of \( G \),

\[
|E(G')| \leq 2|V(G')| - \alpha(G').
\]

To do next: extend proof technique to accommodate reflections
Recall that composition in $\mathbb{R}^2 \rtimes SO(2)$ is given by $(b_1, A_1)(b_2, A_2) = (b_1 + A_1 b_2, A_1 A_2)$. 
Outline of proof

- $S$-symmetry forced rigid graphs are spanning sets in algebraic matroid of $S$-symmetric Cayley-Menger variety
- When $S \subseteq \mathbb{R}^2 \rtimes SO(2)$, this is a Hadamard product of affine spaces
- Each affine space defines two matroids, one which is an elementary lift of the other
- Describe the algebraic matroid of a Hadamard product of affine spaces in terms of these two matroids for each (proof uses tropical geometry)
- Apply to our setting - involves a particular lift of the gain graphic matroid of a complete gain graph
Algebraic matroids

Each subset $S \subseteq E$ defines a coordinate projection $\pi_S : \mathbb{C}^E \to \mathbb{C}^S$.

**Definition**

Let $V \subseteq \mathbb{C}^E$ be an irreducible variety. A given $S \subseteq E$ is

1. **independent** if $\dim(\pi_S(V)) = |S|$,  
2. **spanning** if $\dim(\pi_S(V)) = \dim(V)$, and  
3. a **basis** if $S$ is both independent and spanning.

The common combinatorial structure described by any one of these set systems is called the *algebraic matroid underlying* $V$.  

Let $E = \{1, 2, 3\} \times \{1, 2, 3\}$ and $V \subseteq \mathbb{C}^E$ be the variety of $3 \times 3$ matrices of rank $\leq 1$. Then $S := \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 3)\}$ is spanning, but not independent.

\[
\pi_S(V) = \left\{ \left( \begin{array}{ccc}
 x_{11} & x_{12} & x_{13} \\
 x_{21} & x_{22} & \cdot \\
 \cdot & \cdot & x_{33}
\end{array} \right) : x_{11}x_{22} - x_{21}x_{22} = 0 \right\}
\]
Algebraic matroids in rigidity theory

Definition

Given a pair of integers $d \leq n$, the Cayley-Menger variety of $n$ points in $\mathbb{R}^d$, denoted $CM^d_n$, is the affine variety embedded in $\mathbb{C}^{\binom{n}{2}}$ as the Zariski closure of the set of possible squared pairwise euclidean distances between $n$ points in $\mathbb{R}^d$.

Example

Let $d = 2$. Then the $ij$ coordinate of $CM^2_n$ is parameterized as $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$.

Observation

A graph $G = ([n], E)$ is generically rigid in $\mathbb{R}^d$ if and only if $E$ is spanning in $CM^d_n$. Moreover, $G$ is minimally generically rigid if and only if $E$ is a basis of $CM^d_n$. 
A matroid is a pair \( M = (E, \mathcal{I}) \) where \( E \) is a set and \( \mathcal{I} \subseteq 2^E \) satisfies

1. \( \mathcal{I} \) is nonempty,
2. if \( I \in \mathcal{I} \) and \( J \subseteq I \), then \( J \in \mathcal{I} \), and
3. if \( I, J \in \mathcal{I} \) with \( |I| = |J| + 1 \), then there exists \( e \in I \) such that \( J \cup \{e\} \in \mathcal{I} \).

Elements of \( \mathcal{I} \) are called the independent sets of \( M \).

The rank function \( r_M : 2^E \to \mathbb{Z}_{\geq 0} \) of a matroid \( M = (E, \mathcal{I}) \) maps \( S \subseteq E \) to \( |I| \) where \( I \) is the largest independent subset of \( S \).

A spanning set of \( M = (E, \mathcal{I}) \) is a set \( S \subseteq E \) of maximum rank. A basis is a spanning independent set.
Matroids from submodular functions

Definition (Edmonds 1970)

Let \( f : 2^E \to \mathbb{Z} \) be increasing and submodular, i.e. satisfies:

1. \( f(A) \leq f(B) \) whenever \( A \subseteq B \subseteq E \)
2. \( f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \).

Define \( \mathcal{M}(f) \) to be the matroid on \( E \) where \( I \subseteq E \) is independent iff

\[ \text{for all } I' \subseteq I, \quad I' = \emptyset \quad \text{or} \quad |I'| \leq f(I'). \]

Example (Pym and Perfect 1970)

If \( r_1, \ldots, r_d \) are rank functions of matroids \( M_1, \ldots, M_d \) on ground set \( E \), then \( I \) is independent in \( \mathcal{M}(r_1 + \cdots + r_d) \) iff \( I = I_1 \cup \cdots \cup I_d \) where \( I_j \) is independent in \( M_j \).
Definition

The Hadamard product $u \star v$ of $u, v \in \mathbb{F}^E$ is $(u_e v_e)_{e \in E}$. The Hadamard product of varieties $U, V$ is the Zariski closure of \{ $u \star v : u \in U, v \in V$ \}.

Theorem (DIB 2021)

Let $U, V \subseteq \mathbb{C}^E$ be linear spaces. Then
\[ \mathcal{M}(U \star V) = \mathcal{M}(r_{\mathcal{M}(U)} + r_{\mathcal{M}(V)} - 1). \]

Proposition

$CM^2_n = U \star U$ where $U$ is the linear space spanned by the incidence matrix of the complete graph on $n$ vertices.

Corollary (Lovász and Yemini 1982)

Let $r$ be the rank function of the graphic matroid underlying $K_n$. Then $\mathcal{M}(2r - 1)$ is the algebraic matroid underlying $CM^2_n$.
Symmetric Cayley-Menger varieties

- $S$ is a group of Euclidean isometries of $\mathbb{R}^d$
- $\mathbb{F}^{K_n(S)}$ denotes the $\mathbb{F}$-vector space with coordinates indexed by the arcs of $K_n(S)$
- Define $d : (\mathbb{R}^d)^n \rightarrow \mathbb{R}^{K_n(S)}$ by
  \[
d(z)_e := \| z_{\text{source}(e)} - \text{gain}(e) z_{\text{target}(e)} \|^2_2.
  \]
- $CM^S_n$ is the Zariski closure of the image of $d$
- $S$-gain graph $G$ is generically infinitesimally rigid iff spanning in $CM^S_n$

\[
A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \{A, I\}
\]

\[
d(x_1, y_1, x_2, y_2) = \left( 4x_1^2 + 4y_1^2, (x_1 - x_2)^2 + (y_1 - y_2)^2, \right. \\
\left. (x_1 + x_2)^2 + (y_1 + y_2)^2, 4x_2^2 + 4y_2^2 \right)
\]
Translations and rotations

d = 2 and \( S \) is a subgroup of \( \mathbb{R}^2 \times SO(2) \). If arc \( e \) of \( K_n(S) \) has gain

\[
\left( \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \right)
\]

then under the following change of parameters

\[
x_v \mapsto \frac{x_v + y_v}{2} \quad \quad y_v \mapsto \frac{x_v - y_v}{2i}
\]

the entry of \( CM^S_n \) corresponding to \( e \) is

\[
(x_{\text{source}}(e) - e^{i\theta} x_{\text{target}}(e) - a - bi)(y_{\text{source}}(e) - e^{-i\theta} y_{\text{target}}(e) - a + bi)
\]

and so \( CM^S_n \) is a Hadamard product of affine spaces.

This part gets more complicated, though not hopeless, when \( S \) has reflections.
Let $V = \{Ax + b : x \in \mathbb{C}^d\} \subseteq \mathbb{C}^E$ be an affine space.

- the algebraic matroid $\mathcal{M}(V)$ of $V$ is the row matroid of $A$
- define $\mathcal{M}^L(V)$ to be the row matroid of $(A \ b)$
- $\mathcal{M}^L(V)$ is an elementary lift of $\mathcal{M}(V)$
- $I \subseteq E$ is independent in $\mathcal{M}(V)$ implies $I$ independent in $\mathcal{M}^L(V)$

**Theorem (DIB 2021)**

Let $U, V \subseteq \mathbb{C}^E$ be finite-dimensional affine spaces and define $f : 2^E \rightarrow \mathbb{Z}$ by

$$f(S) = \begin{cases} 
    r_{\mathcal{M}(U)}(S) + r_{\mathcal{M}(V)}(S) & \text{if } r_{\mathcal{M}(U)}(S) < r_{\mathcal{M}^L(U)}(S) \\
    r_{\mathcal{M}(V)}(S) & \text{or } r_{\mathcal{M}(V)}(S) < r_{\mathcal{M}^L(V)}(S) \\
    r_{\mathcal{M}(U)}(S) + r_{\mathcal{M}(V)}(S) - 1 & \text{otherwise.}
\end{cases}$$

Then $\mathcal{M}(U \star V) = \mathcal{M}(f)$. 

Daniel Irving Bernstein (MIT and Fields)  
Symmetry-forced rigidity in the plane
Gain graphic matroids

**Definition**

The *gain-graphic matroid* of a gain graph $G$ is the matroid supported on the arc set of $G$ whose independent sets are sets of arcs such that each connected component has at most one cycle, which is not balanced.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S = \{I, A, A^2, A^3\}$$

**Example**

In the above gain graph, adding the loop to any spanning tree produces a basis of the underlying gain-graphic matroid.
Putting it all together

If \( S \subseteq \mathbb{R}^2 \rtimes SO(2) \), \( CM_n^S = U \star V \) where \( U, V \) are affine spaces satisfying

- \( M(U) = M(V) \) is the gain-graphic matroid of the gain graph obtained from \( K_n(S) \) by ignoring the translation part of each gain
- \( M^L(U) = M^L(V) \) is obtained from the gain graph of \( K_n(S) \) by making non-Dutch bicyclic subgraphs independent

**Theorem (DIB 2021)**

Let \( S \) be a subgroup of \( \mathbb{R}^2 \rtimes SO(2) \). For each \( S \)-gain graph \( H \), define

\[
\alpha(H) = \begin{cases} 
3 & \text{if every cycle in } H \text{ is balanced} \\
2 & \text{if not, and the gain of each cycle is a translation} \\
1 & \text{if none of the above, and all bicyclic subgraphs are Dutch} \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( G \) is independent in \( M(CM_n^S) \) if and only if

\[
|E(G')| \leq 2|V(G')| - \alpha(G')
\]

for all subgraphs \( G' \) of \( G \).
Thank you for your attention!