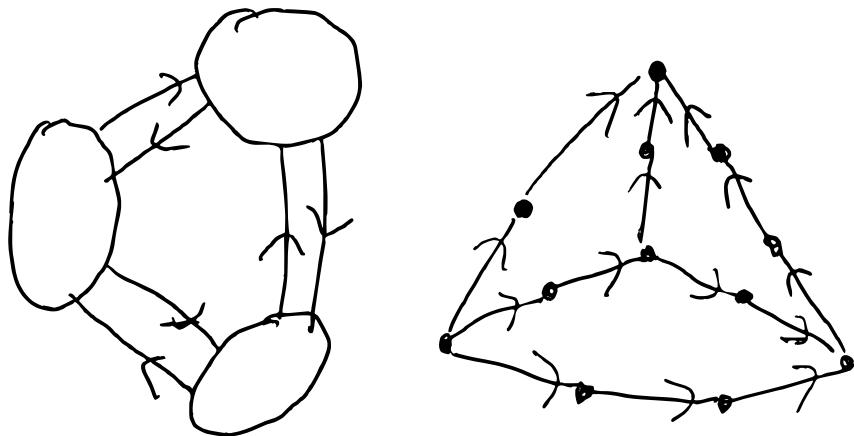


# Digraphs: Substructures and Colorings



Raphael Steiner (TU Berlin)

contains joint works with

Lior Gishboliner, Tamás Mészáros,  
Tibor Szabó

# Three famous problems on graph coloring and substructures

WHICH STRUCTURES CAN BE FOUND  
IN GRAPHS OF LARGE  $\chi$ ?

Hadwiger's Conjecture (1943):

If  $\chi(G) \geq t$ , then  $G \supseteq K_t$ .

Hajós' Conjecture (1950):

If  $\chi(G) \geq t$ , then  $G \supseteq K_t$ .

Gyárfás-Sumner-Conjecture (1975):

For every forest  $F$  and  $k \in \mathbb{N}$  there is  $c(F, k) \in \mathbb{N}$  s.t. if  $\chi(G) \geq c(F, k)$ ,  
then  $G \supseteq_{\text{ind}} F$  or  $G \supseteq_{\text{ind}} K_k$ .

WHICH STRUCTURES CAN BE FOUND  
IN GRAPHS OF LARGE  $\chi$ ?



Observation: For every  $G$  there is  $H \subseteq G$   
s.t.  $\delta(H) \geq \chi(G) - 1$ .



WHICH STRUCTURES CAN BE FOUND  
IN GRAPHS OF LARGE  $\delta$ ?

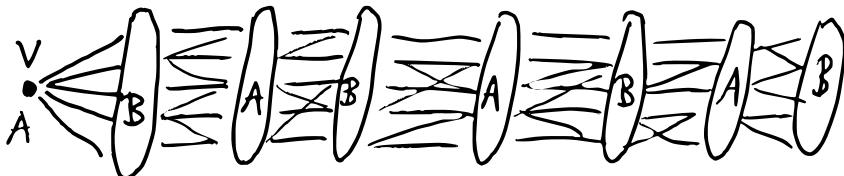


WHICH STRUCTURES CAN BE FOUND  
IN GRAPHS OF LARGE DENSITY/  
AVERAGE DEGREE?

# History of Hajwigert Hajós

**Wagner ('64)** If  $\chi(G) > 2^{t-2}$ , then  $G \triangleright K_t$ .

**Jung ('65)** If  $\chi(G) > 2^{t^{2/2}}$ , then  $G \not\triangleright K_t$ .



**Mader ('67)** If  $\delta(G) \geq 2^{t-2}$ , then  $G \triangleright K_t$ .

If  $\delta(G) \geq 2^{t^{2/2}}$ , then  $G \not\triangleright K_t$ .

**Catlin ('79)** Hajós' Conjecture is wrong for  $t \geq 7$ .

**Erdős + Fajtlowicz ('81)** Hajós' Conjecture is wrong for random graphs.

Lower bound: There exist  $K_t$ -subdivision-free graphs  $G$  with  $\chi(G) = \Omega\left(\frac{t^2}{\log t}\right)$ .

# History of Hadwiger + Hajós

Kostochka ('84)  
Thomason

If  $\delta(G) \geq c \cdot t \cdot \sqrt{\log t}$ ,  
then  $G \not\simeq K_t$  (tight).

Bollobás + Thomason ('98)  
Komlós + Szemerédi ('96)

If  $\delta(G) \geq c \cdot t^2$ ,  
then  $G \not\simeq K_t$  (tight).

Norine +  
Postle + ('19-20)  
Song

If  $\chi(G) \geq c \cdot t \cdot (\log t)^\beta$ ,  
then  $G \not\simeq K_t$   
(for any  $\beta > 1/4$ ) .

Postle ('20)

If  $\chi(G) \geq c \cdot t \cdot (\log \log t)^6$ ,  
then  $G \not\simeq K_t$ .

# What about digraphs?

## 1. WHAT IS THE CHROMATIC NUMBER?

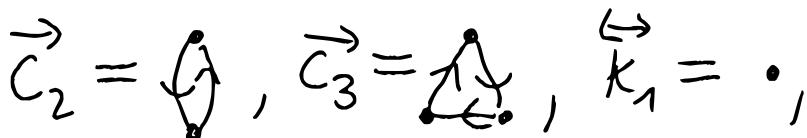
Erdős ('79),

Neumann-Lara ('80)

A coloring of a digraph  
is a vertex-coloring  
without monochromatic cycles.

↳ DICHROMATIC NUMBER  $\vec{\chi}(\mathcal{D})$

Examples:  $\vec{\chi}(\vec{C}_\ell) = 2$ ,  $\vec{\chi}(\vec{K}_t) = t$



But  $\vec{\chi}(\vec{K}_4) = \vec{\chi}(\text{graph with 4 nodes in a circle with edges } (1,2), (2,3), (3,4), (4,1)) = 1$ .

## 2. WHAT ARE MINORS AND SUBDIVISIONS?

(A) Weak minors (Jagger '96):

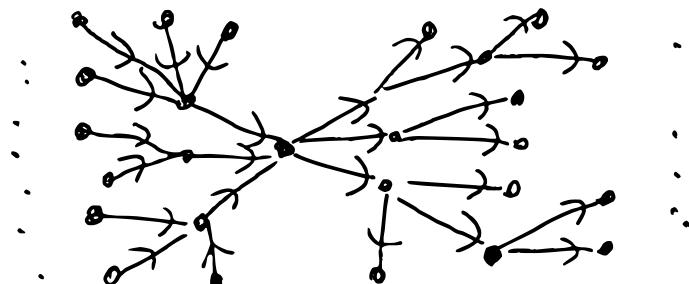
Take subgraphs + Contract edges /  
weakly connected subgraphs

(B) Strong minors (Jagger '96, Kim+ Seymour '15,  
Axenovich, Girão, Snyder, Weber '20)

Take subgraphs + Contract directed cycles/  
strongly connected subgraphs

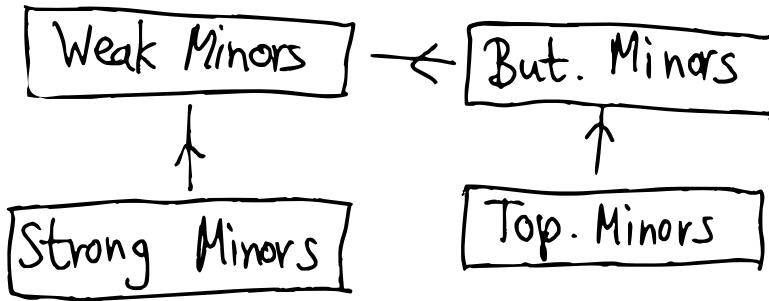
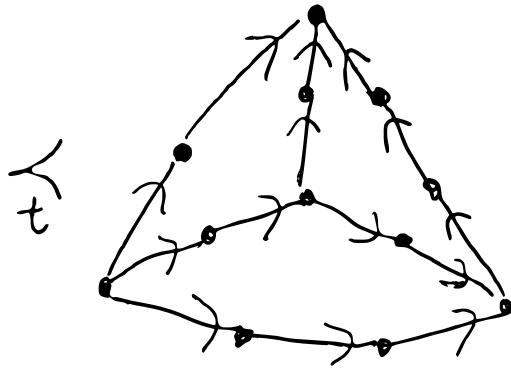
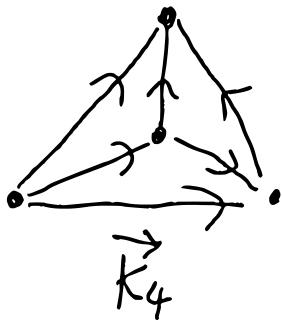
(C) Butterfly minors (Johnson, Robertson, Seymour,  
Thomas '01,  
Kawarabayashi+Kreutzer)

Take subgraphs + Contract "butterflies":



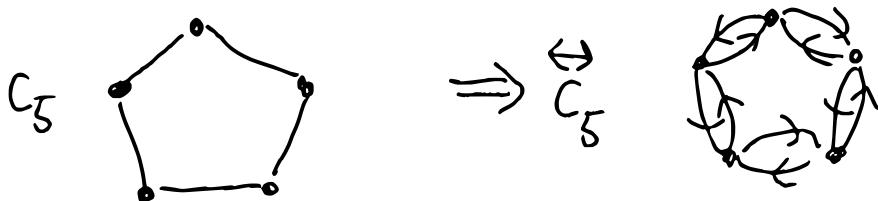
## 2. WHAT ARE MINORS AND SUBDIVISIONS?

(D) Topological minors / Subdivisions:



### 3. HOW DO THESE DEFINITIONS GENERALIZE UNDIRECTED ONES?

$G$  undirected  $\Rightarrow \overset{\leftrightarrow}{G}$  "bi orientation"



1.  $\chi(\overset{\leftrightarrow}{G}) = \chi(G)$

2.  $G \succ H$  iff  $\overset{\leftrightarrow}{G}$  contains  $\overset{\leftrightarrow}{H}$  as weak / strong / but.-minor

$G \succ_t H$  iff  $\overset{\leftrightarrow}{G} \succ_t \overset{\leftrightarrow}{H}$ .

WHICH STRUCTURES CAN BE FOUND  
IN DIGRAPHS OF LARGE  $\vec{\chi}$ ?



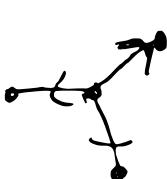
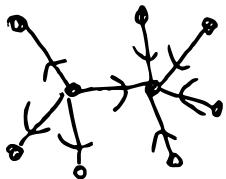
Observation: For every  $D$  there is  $H \subseteq D$   
s.t.  $\delta^+(H), \delta^-(H) \geq \vec{\chi}(D) - 1$ .



WHICH STRUCTURES CAN BE FOUND  
IN DIGRAPHS OF LARGE  $\delta^+/\delta^-$ ?

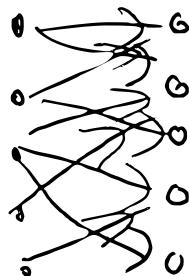
WHICH STRUCTURES CAN BE FOUND  
IN DIGRAPHS OF LARGE DENSITY/  
AVERAGE DEGREE?

Density is not really interesting



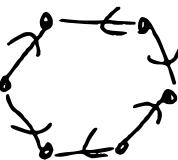
**Observation** Unless  $F$  is an antidirected forest, there are digraphs of large density not containing  $F$  as a strong/butterfly/topological minor:

- bipartite
- large density
- large girth



no  $\rightarrow \bullet \rightarrow \bullet$

, no



Jagger '96

If the average degree is at least  $C \cdot t \cdot \sqrt{\log t}$ , then  $D$  contains  $K_t$  as a weak minor.

# Digraphs of large minimum degree

Thomassen '85

For every  $k \in \mathbb{N}$   $\exists D_k$  s.t.

$\delta^+(D_k), \delta^-(D_k) \geq k$  and  
all dicycles in  $D_k$  are even.

Cor

$D_k$  does not contain  $\overset{\leftrightarrow}{K}_3$



as a butterfly- / topological minor.

Axenovich et al. '12 0

$D_k$  does not contain  $\overset{\leftrightarrow}{K}_3$   
as a strong minor.

Thomassen '85

Digraphs of large  $\delta^+$  need  
not contain

"need not contain



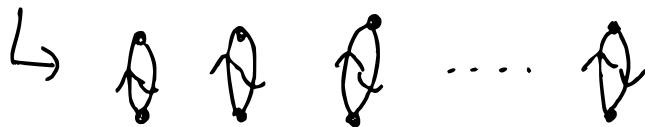
De Vos, McDonald,  
Mohar, Scheide '11

# Digraphs of large minimum degree

Thomassen '83

If  $\delta^+(\mathcal{D}) \geq (k+1)!$ , then  $\mathcal{D}$

contains  $k$  disjoint dicycles.



Bermond-Thomassen-Conjecture '81:

If  $\delta^+(\mathcal{D}) \geq 2k-1$ , then  $\mathcal{D}$  has  $k$  disjoint cycles.

$k=4$  still open, best bound:  $18k$  Bucić '18

Mader's Conjecture '85:

For every acyclic  $F$  there exists  $d(F) \in \mathbb{N}$  s.t.

$$\delta^+(\mathcal{D}) \geq d(F) \Rightarrow \mathcal{D} \not\cong F.$$

Mader '96

$$d(\vec{K}_4) = 3 ; d(\vec{K}_5) < \infty ?$$

# Digraphs of large minimum degree

Aboulker, Cohen, Havet, Locket, Moura, Thomassé '19

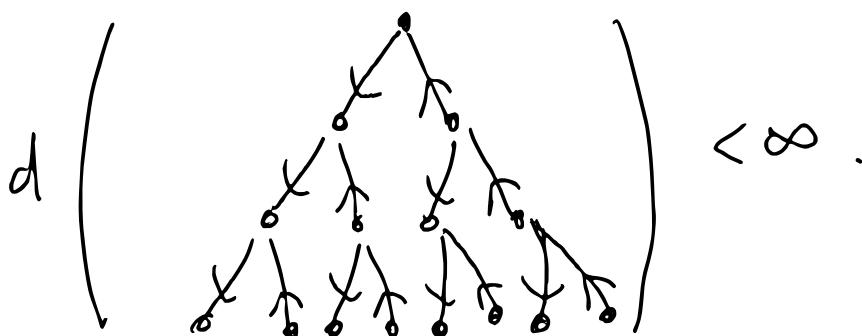
Conjecture 1:  $d(F)$  exists if  $F$  is an orientation of a forest.

Conjecture 2:  $d(F)$  exists if  $F$  is an orientation of a cycle.

Gishboliner, S,  
Szabó '20

Conjecture 2 is true, also for  
disjoint unions of cycles.

Not known whether



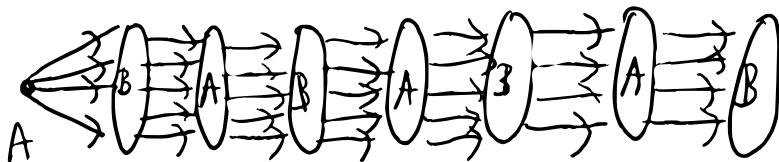
# Hadwiger - and Hajós - type problems for digraphs

Let  $F$  be fixed. Do the digraphs excluding  $F$  as a weak/strong/butterfly/topological minor have a bounded dichromatic number?

Aboulker, Cohen, Havet, Lachet, Moura, Thomassé ('16)

Yes, for topological/butterfly minors:

If  $\vec{\chi}(D) \geq 4^{t^2-2t+1} (t-1) + 1$ , then  
 $D$  contains a  $\overleftrightarrow{K}_t$ -subdivision.



Axenovich, Girão, Snyder, Weber '20

Yes, for strong minors/weak minors:

If  $\vec{\chi}(D) \geq 4^t \cdot t$ , then  $D \triangleright \overleftrightarrow{K}_t$ .

# Hadwiger - and Hajós - type problems for digraphs

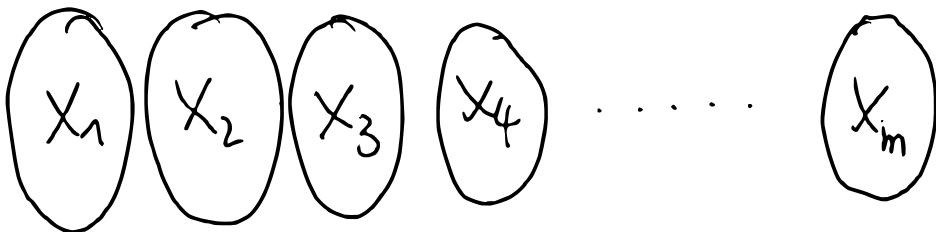
Mészáros, S'21 If  $\vec{\chi}(\mathcal{D}) \geq c \cdot t \cdot (\log \log t)^6$ ,  
then  $\mathcal{D} \gtrsim_s^{\leftrightarrow} K_t$ ,  $\mathcal{D} \gtrsim_b^{\leftrightarrow} K_t$ .

Proof sketch:

Claim: For every  $\mathcal{D}$  there is  $G$  s.t.

- $\vec{\chi}(\mathcal{D}) \leq 2\chi(G)$ ,
- $\mathcal{D} \gtrsim_s G$

$X_i \subseteq V(\mathcal{D}) \setminus (X_1 \cup X_2 \cup \dots \cup X_{i-1})$  largest s.t.  
 $\mathcal{D}[X_i]$  strong + 2-colorable



# Hadwiger - and Hajós - type problems for digraphs

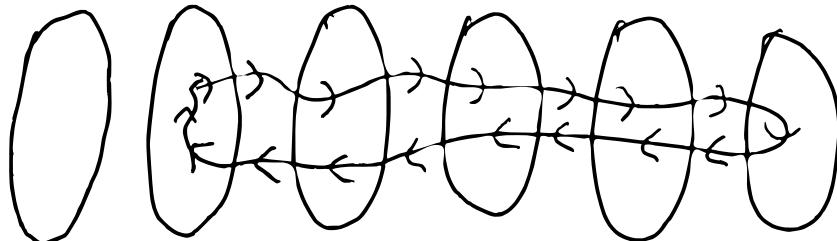
$$V(G) = \{1, \dots, m\},$$

$$ij \in E(G) : \Leftrightarrow \begin{matrix} & \\ \text{---} & \end{matrix} \begin{matrix} x_i \\ \leftarrow \rightarrow \\ x_j \end{matrix}$$

$$\hookrightarrow D \not\sim G.$$

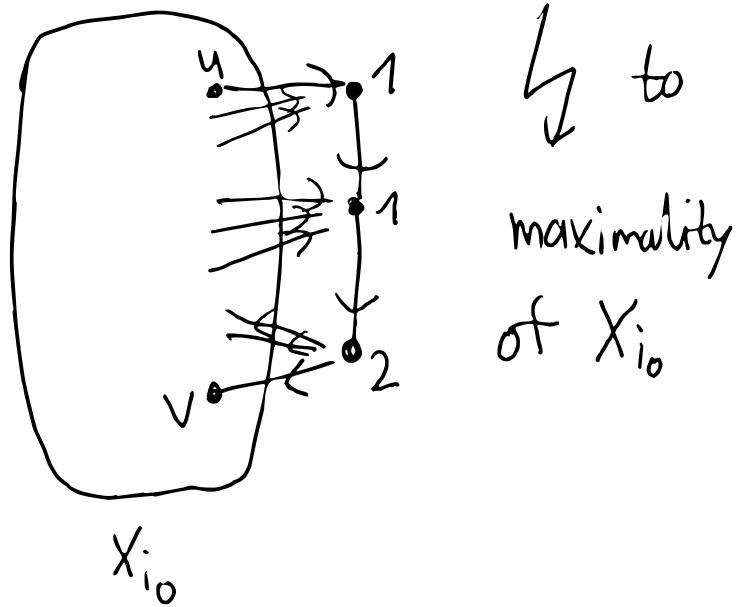
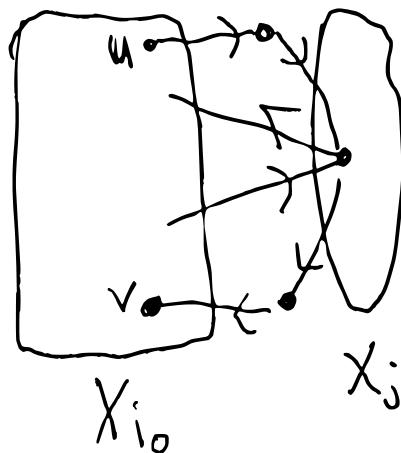
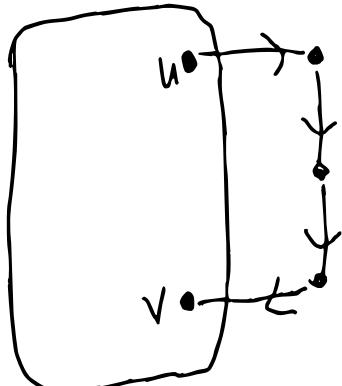
$f: V(G) \rightarrow \{c_1, \dots, c_k\}$  optimal coloring.

Color  $x_i$  with colors  $c_i^{(1)}, c_i^{(2)}$  acyclically.



$i_0$   
 $\downarrow$   
leftmost

Not possible:



Proof conclusion:

$$\mathcal{D} \not\propto R_t \Rightarrow \overleftarrow{G} \not\propto K_t \Rightarrow G \not\propto K_t$$
$$\Rightarrow \vec{x}(\mathcal{D}) \leq 2x(G) \leq O(t(\log \log t)^6) \quad \square$$

# Results for subdivisions

**Aboulker et al. '16** If  $\vec{x}(D) \geq 4^{\alpha(F)} \cdot v(F)$ , then  $D \not\geq F$ .

Conjecture: If  $\vec{x}(D) \geq v(F)$ , then  $D \not\geq F$   
for  $F$  orientation of a cycle.

**Gishboliner, S., Szabó '20** If  $\vec{x}(D) \geq v(F)$  and  $F$  is an oriented cactus, then  $D \not\geq F$ .

**Dirac '52** If  $x(G) \geq 4$ , then  $G \not\geq K_4$ .

**Gishboliner, S., Szabó '20** If  $\vec{x}(D) \geq 4$ , then  $D \not\geq T$  for every tournament  $T$  of order 4.

**Mészáros, S. '21** If  $\vec{x}(D) \geq 22 v(F)$  and  $F$  is subcubic, then  $D \not\geq F$ .

**Open Problem** Is the dichromatic number of  $K_t$ -subdivision-free digraphs polynomial in  $t$ ?

# Forbidden induced subgraphs

Question:  $\mathcal{F}$  finite. When does  
 $\text{Forbind}(\mathcal{F})$  have bounded  
dichromatic number?  
↳ "heroic" set.

Conjecture (Aboulker, Chabot, Naserasr '20):

1.  $\mathcal{F}$  oriented forest,  $k \in \mathbb{N}$ . Then

$\{\overleftrightarrow{K}_2, \mathcal{F}, \vec{K}_k\}$  is heroic.

2.  $\mathcal{F}$  oriented starforest,  $H$  hero:

$\{\overleftrightarrow{K}_2, \mathcal{F}, H\}$  is heroic.

nice open  
problem

Does an oriented graph of huge  $\vec{\chi}$   
necessarily contain a large clique or a  
long induced directed path?

Gyárfás '87

If  $\chi(G)$  is sufficiently large,  
then  $G$  contains  $P_k$  or  $K_k$   
as an induced subgraph.



