Digraphs: Substructures and Colorings

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contains joint works with
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Three famous problems on graph coloring and substructures

Which structures can be found in graphs of large \( x \)?

**Hadwiger's Conjecture (1943):**

If \( \chi(G) \geq t \), then \( G \cong K_t \).

**Hajós' Conjecture (1950):**

If \( \chi(G) \geq t \), then \( G \cong K_t \).

**Gyárfás-Sumner Conjecture (1975):**

For every forest \( F \) and \( k \in \mathbb{N} \) there is \( c(F, k) \in \mathbb{N} \) s.t.

if \( \chi(G) \geq c(F, k) \), then \( G \cong F \) or \( G \cong K_k \).
WHICH STRUCTURES CAN BE FOUND IN GRAPHS OF LARGE $\chi$?

\[ \downarrow \]

Observation: For every $G$, there is $H \subseteq G$ s.t. $\delta(H) \geq \chi(G) - 1$.

\[ \downarrow \]

WHICH STRUCTURES CAN BE FOUND IN GRAPHS OF LARGE $\delta$?

\[ \uparrow \]

WHICH STRUCTURES CAN BE FOUND IN GRAPHS OF LARGE DENSITY/AVERAGE DEGREE?
History of Hadwiger–Hajós

**Wagner (1964)** If $\chi(G) > 2^{t-2}$, then $G \nsubseteq K_t$.

**Jung (1965)** If $\chi(G) > 2^{t^{3/2}}$, then $G \nsubseteq K_t$.

**Mader (1967)**
- If $\delta(G) \geq 2^{t-2}$, then $G \nsubseteq K_t$.
- If $\delta(G) \geq 2^{t^{3/2}}$, then $G \nsubseteq K_t$.

**Catlin (1979)** Hajós’ Conjecture is wrong for $t \geq 7$.

**Erdős + Fajtlowicz (1981)**
- Hajós’ Conjecture is wrong for random graphs.
- Lower bound: There exist $K_t$-subdivision-free graphs $G$ with $\chi(G) = \Omega \left( \frac{t^2}{\log t} \right)$. 
History of Hadwiger-Hajós

Kostochka (’84)  
Thomason

If \( d(G) \geq C \cdot t \cdot \sqrt{\log t} \), then \( G \geq K_t \) (tight).

Bollobás+Thomason (’81)  
Komlós+Szemerédi (’86)

If \( d(G) \geq C \cdot t^2 \), then \( G \geq K_t \) (tight).

Norine+Postle+ (’19-20)  
Song

If \( x(G) \geq C \cdot t \cdot (\log t)^{\beta} \), then \( G \geq K_t \) (for any \( \beta > 1/4 \)).

Postle (’20)

If \( x(G) \geq C \cdot t \cdot (\log \log t)^6 \), then \( G \geq K_t \).
What about digraphs?

1. WHAT IS THE CHROMATIC NUMBER?

Erdős ('79), Neumann-Lara ('80)

A coloring of a digraph is a vertex-coloring without monochromatic cycles.

\[ \text{Dichromatic Number } \vec{\chi}(D) \]

Examples: \[ \vec{\chi}(\vec{C}_2) = 2, \quad \vec{\chi}(\vec{K}_t) = t \]

\[ \vec{C}_2 = \{ \cdot \}, \quad \vec{C}_3 = \{ \cdot \to \uparrow \to \cdot \}, \quad \vec{K}_1 = \cdot, \]
\[ \vec{K}_2 = \vec{C}_2, \quad \vec{K}_3 = \{ \cdot \to \cdot \to \cdot \}, \quad \vec{K}_4 = \]

But \[ \vec{\chi}(\vec{K}_4) = \chi(\text{ } \text{ } \text{ } \text{ }) = 1. \]
2. WHAT ARE MINORS AND SUBDIVISIONS?

(A) Weak minors (Jagger '96):
Take subgraphs + Contract edges / weakly connected subgraphs

(B) Strong minors (Jagger '96, Kim+Seymour '15, Axenovich, Girão, Snyder, Weber '20)
Take subgraphs + Contract directed cycles / strongly connected subgraphs

(C) Butterfly minors (Johnson, Robertson, Seymour, Thomas '01, Kawarabayashi+Kreutzer)
Take subgraphs + Contract "butterflies":
2. WHAT ARE MINORS AND SUBDIVISIONS?

(D) Topological minors / Subdivisions:

\[ K_4 \]

Weak Minors  \xrightarrow{t}  But. Minors

Strong Minors  \xrightarrow{t}  Top. Minors
3. How do these definitions generalize undirected ones?

\[ G \text{ undirected} \Rightarrow \overset{\leftrightarrow}{G} \text{ "bi-orientation"} \]

1. \[ \chi(\overset{\leftrightarrow}{C_5}) = \chi(G) \]

2. \[ G \succeq H \text{ iff } \overset{\leftrightarrow}{G} \text{ contains } \overset{\leftrightarrow}{H} \text{ as weak/strong/but.-minor} \]

\[ G \succeq_t H \text{ iff } \overset{\leftrightarrow}{G} \succeq_t \overset{\leftrightarrow}{H} \]
WHICH STRUCTURES CAN BE FOUND IN DIGRAPHS OF LARGE $\overrightarrow{\chi}$? \\
\downarrow \\
Observation: For every $D$ there is $H \subseteq D$ s.t. $\delta^+(H), \delta^-(H) \geq \overrightarrow{\chi}(D) - 1$. \\
\downarrow \\
WHICH STRUCTURES CAN BE FOUND IN DIGRAPHS OF LARGE $\delta^+/\delta^-$? \\

WHICH STRUCTURES CAN BE FOUND IN DIGRAPHS OF LARGE DENSITY/AVERAGE DEGREE?
Density is not really interesting

Observation: Unless $F$ is an antidirected forest, there are digraphs of large density not containing $F$ as a strong/butterfly/topological minor:

- bipartite
- large density
- large girth

Jagger '96

If the average degree is at least $C \cdot t \cdot \sqrt{\log t}$, then $D$ contains $K_t$ as a weak minor.
Digraphs of large minimum degree

Thomassen '85

For every $k \in \mathbb{N}$, $D_k$ s.t.

$\delta^+(D_k), \delta^-(D_k) \geq k$ and

all dicycles in $D_k$ are even.

Cor

$D_k$ does not contain $K_3 = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}$

as a butterfly-/topological minor.

Axenovich et al. 120

$D_k$ does not contain $K_3$

as a strong minor.

Thomassen '85

Digraphs of large $\delta^+$ need not contain $\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}$ as a topological minor.

De Vos, Mcdonald, Mohar, Scheide '11

need not contain $\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}$
Digraphs of large minimum degree

Thomassen '83: If \( \delta^+(D) \geq (k+1)! \), then \( D \) contains \( k \) disjoint cycles.

\[
\Rightarrow \quad \bullet \quad \bullet \quad \bullet \quad \ldots \quad \bullet
\]

Bermond-Thomassen-Conjecture '81:
If \( \delta^+(D) \geq 2k-1 \), then \( D \) has \( k \) disjoint cycles.

\( k = 4 \) still open, best bound: \( 18k \) Bucić '98

Mader's Conjecture '85:
For every acyclic \( F \) there exists \( d(F) \in \mathbb{N} \) s.t.
\[
\delta^+(D) \geq d(F) \Rightarrow D \not\rightarrow F.
\]

Mader '96: \( d(K_4^2) = 3 \); \( d(K_5^2) < \infty \)?
Digraphs of large minimum degree
Aboulker, Cohan, Havet, Lochet, Mowla, Thomassé ’19

**Conjecture 1:** \(d(F)\) exists if \(F\) is an orientation of a forest.

**Conjecture 2:** \(d(F)\) exists if \(F\) is an orientation of a cycle.

Gishboliner, S, Szabó ’20

Conjecture 2 is true, also for disjoint unions of cycles.

Not known whether

\[
d\left(\begin{array}{c}
& \vdots \\
& \vdots \\
& \vdots \\
\end{array}\right) < \infty
\]
Hadwiger- and Hajós-type problems for digraphs

Let $F$ be fixed. Do the digraphs excluding $F$ as a weak/strong/butterfly/topological minor have a bounded dichromatic number?

Aboulker, Cohen, Havet, Lachet, Mouri, Thomassé (176)

Yes, for topological/butterfly minors:

If $\chi(D) \geq 4^{t^2-2t+1}(t-1)+1$, then $D$ contains a $K_t$-subdivision.

Axenovich, Girão, Snyder, Wêker '20

Yes, for strong minors/weak minors:

If $\chi'(D) \geq 4^t \cdot t$, then $D \not\cong K_t$.
**Hadwiger- and Hajós-type problems for digraphs**

Mészáros, S.'21

If \( \chi(D) \geq C \cdot t \cdot (\log \log t)^6 \), then \( D \cong R_t \), \( D \supseteq R_t \).

**Proof sketch:**

**Claim:** For every \( D \) there is \( G \), s.t.
- \( \chi(D) \leq 2 \chi(G) \),
- \( D \cong G \)

\( X_i \subseteq V(D) \setminus (X_1 \cup X_2 \cup \cdots \cup X_{i-1}) \) (largest s.t. \( D[X_i] \) strong + 2-colorable)
Hadwiger- and Hajós-type problems for digraphs

\[ V(\delta) = \{1, \ldots, m\}, \]

\[ ij \in E(\delta) \iff X_i \rightarrow X_j \]

\[ G_1 \supset G_0 \]

\[ f : V(\delta) \rightarrow \{c_1, \ldots, c_k\} \text{ optimal coloring.} \]

Color \( X_i \) with colors \( c_i^{(1)}, c_i^{(2)} \) acyclically.

\[ \text{Leftmost} \]
Proof conclusion:

\[ D \times_s K_t \Rightarrow G \times_s K_t \Rightarrow G \times K_t \]

\[ \Rightarrow \chi(D) \leq 2\chi(G) = O(t (\log \log t)^6) \]
Results for subdivisions

Aboulker et al. '16  If $\chi^*(D) \geq 4 \alpha(F) \cdot v(F)$, then $D \geq_\ell F$.

Conjecture:  If $\chi^*(D) = v(F)$, then $D \geq_\ell F$ for $F$ orientation of a cycle.

Gishboliner, S., Szabó '20  If $\chi^*(D) \geq v(F)$ and $F$ is an oriented cactus, then $D \geq_\ell F$.

Dirac '52  If $\chi(G) \geq 4$, then $G \geq_\ell K_4$.

Gishboliner, S., Szabó '20  If $\chi^*(D) \geq 4$, then $D \geq_\ell T$ for every tournament $T$ of order 4.

Mészáros, S. '21  If $\chi^*(D) \geq 22 \alpha(F)$ and $F$ is subcubic, then $D \geq_\ell F$.

Open Problem  Is the dichromatic number of $K_t$-subdivision-free digraphs polynomial in $t$2?
Forbidden induced subgraphs

**Question:** $\mathcal{F}$ finite. When does $
\text{Forbind } (\mathcal{F}) \text{ have bounded dichromatic number?}$

$\rightarrow$ "heroic" set.

**Conjecture (Aboulker, Charbit, Naserasr '20):**

1. $\mathcal{F}$ oriented forest, $k \in \mathbb{N}$. Then
   $\{\vec{K}_2, \vec{F}, \vec{K}_k\}$ is heroic.
2. $\mathcal{F}$ oriented star forest, $H$ hero:
   $\{\vec{K}_2, \vec{F}, H\}$ is heroic.

Does an oriented graph of huge $\vec{X}$ necessarily contain a large clique or a long induced directed path?
Gyárfás '87  If \( \chi(G) \) is sufficiently large, then \( G \) contains \( P_k \) or \( K_k \) as an induced subgraph.

Thank you!