# Two open problems

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Polynomial Sphere Recognition

# Statement

## Polynomial Sphere Recognition (geometric version)

Is there a polynomial algorithms that decides whether a triangulated 3-manifold is isomorphic to the 3-sphere?

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Is there a polynomial algorithms that decides whether a 2-complex without holes (that is $H_1 = 0$) embeds in 3-sphere?

# Background

Geometric approach:

1. Topological, differentiable and PL embedding are equivalent for this question (1950s);
2. Perelman's Theorem (2006)
3. Problem is in co-NP and NP (Heusner, Zentner, Ivanov, Schleimer 2001–2016)
4. Curvature of 2-complexes?
5. Discrete Ricci-flow?

# Background

Combinatorial approach:

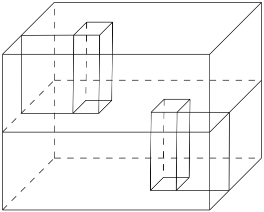
1. Without assumption $H_1 = 0$ problem gets NP-hard (de Mesmay, Rieck, Sedgewick, Tancer 2017);
2. quadratic algorithm in the simply connected case (C $2019^+$);
3. it is undecidable whether a 2-complex is simply connected.
4. ..

# Why is it so hard?

**Fact**

The following are equivalent for a graph $G$

- $G$ is a tree;
- $G$ is simply connected;
- $G$ has no holes ($H_1 = 0$).



# Trying to understand the difficulties

**Theorem (C, Lichev 2020$^+$)**

There is a simply connected 2-complex $C$ that embeds in 3-space such that $C/e$ does not embed for every edge $e$.
The reason is that any edge is a loop that must be embedded in a knotted way.

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# Open problem 2

Matroid version of the local Tutte Theorem

## Theorem (C $2020^+$)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.
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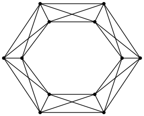

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- Matroids?

# Statement of the problem

**Theorem 1 (C $2020^+$)**

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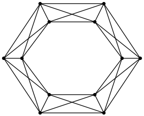

**Open problem**

Prove a matroid analogue of Theorem 1!