Two open problems

Johannes Carmesin

University of Birmingham

Matroid Union Seminar, December 14, 2020
Open problem 1

Polynomial Sphere Recognition
Polynomial Sphere Recognition (geometric version)

Is there a polynomial algorithm that decides whether a triangulated 3-manifold is isomorphic to the 3-sphere?

Polynomial Sphere Recognition (combinatorial version)
Statement

Polynomial Sphere Recognition (geometric version)
Is there a polynomial algorithms that decides whether a triangulated 3-manifold is isomorphic to the 3-sphere?

Polynomial Sphere Recognition (combinatorial version)
Is there a polynomial algorithms that decides whether a 2-complex without holes (that is $\mathcal{H}_1 = 0$) embeds in 3-sphere?
Geometric approach:

1. Topological, differentiable and PL embedding are equivalent for this question (1950s);
2. Perelman’s Theorem (2006)
3. Problem is in co-NP and NP (Heusner, Zentner, Ivanov, Schleimer 2001–2016)
4. Curvature of 2-complexes?
5. Discrete Ricci-flow?
Background

Combinatorial approach:

1. Without assumption $H_1 = 0$ problem gets NP-hard (de Mesmay, Rieck, Sedgewick, Tancer 2017);
2. quadratic algorithm in the simply connected case (C 2019$^+$);
3. it is undecidable whether a 2-complex is simply connected.
4. ..
Fact

The following are equivalent for a graph $G$

1. $G$ is a tree;
2. $G$ is simply connected;
3. $G$ has no holes ($H_1 = 0$).
Trying to understand the difficulties

Theorem (C, Lichev 2020+)
There is a simply connected 2-complex $C$ that embeds in 3-space such that $C/e$ does not embed for every edge $e$. The reason is that any edge is a loop that must be embedded in a knotted way.
Statement

**Polynomial Sphere Recognition (geometric version)**
Is there a polynomial algorithms that decides whether a triangulated 3-manifold is isomorphic to the 3-sphere?

**Polynomial Sphere Recognition (combinatorial version)**
Is there a polynomial algorithms that decides whether a 2-complex without holes (that is \( H_1 = 0 \)) embeds in 3-sphere?
Matroid version of the local Tutte Theorem
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

What are local 2-separators?
✓ Exact characterisation of graph with no bounded subdivision of a wheel;
✓ Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation);
✓ Local version of Tangle-Tree Theorem
✓ Local Grid Theorem
?
Theorem (C 2020+) 

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$. 

+ Uniqueness statement

- What are local 2-separators? ✓
Theorem (C 2020$^+$)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
- Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation); ✓
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
- Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation); ✓
- Local version of Tangle-Tree Theorem ✓
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
- Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation); ✓
- Local version of Tangle-Tree Theorem ✓
- Local Grid Theorem ?
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
- Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation); ✓
- Local version of Tangle-Tree Theorem ✓
- Local Grid Theorem ?
- Structure Theorem for graphs without a bounded $K_r$-minor ???
Theorem (C 2020+)

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

- What are local 2-separators? ✓
- Exact characterisation of graph with no bounded subdivision of a wheel; ✓
- Embedding graphs in surfaces maximally locally planar (Whitney-type characterisation); ✓
- Local version of Tangle-Tree Theorem ✓
- Local Grid Theorem ?
- Structure Theorem for graphs without a bounded $K_r$-minor ???
- Matroids?

Johannes Carmesin

Two open problems
Theorem 1 (C 2020\textsuperscript{+})

For every $r$, every connected $r$-locally 2-connected graph $G$ has a graph-decomposition of adhesion two and locality $r$ such that all its torsos are $r$-locally 3-connected or cycles of length at most $r$.

+ Uniqueness statement

Open problem

Prove a matroid analogue of Theorem 1!