Decomposing sparse graphs into pseudoforests

Definition

A *pseudoforest* is a graph where every connected component contains at most one cycle.

Definition

For integers k and d, a graph G is (k, d)*-decomposable* if we can partition the edge set into k pseudoforests where one of the pseudoforests has maximum degree d. Similarly, a graph is $(k, d)^*$*-decomposable* if we can partition the edge set into k pseudoforests where one of the pseudoforests has every connected component containing at most d edges.

A Nine Dragon Tree type problem on pseudoforests

Theorem (Genghua Fan, Yan Li, Ning Song, and Daqing Yang 2015)

Let k and d be integers. Every graph with maximum average degree at most $2k + \frac{2d}{k+d+1}$ is $(k+1, d)$-decomposable. Further, for every k, d and every $\varepsilon > 0$, there exists a graph with maximum average degree at most $2k + \frac{2d}{k+d+1} + \varepsilon$ which is not $(k+1, d)$-decomposable.

Conjecture

Let k and d be integers. There exists a $\varepsilon > 0$ and an integer $g \geq 5$ such that every graph with girth g and maximum average degree at most $2k + \frac{2d}{k+d+1} + \varepsilon$ is $(k+1, d)$-decomposable. (Bonus points if you get a $(k+1, d)^*$-decomposition).