Asymptotic dimension

Let G be a graph.

The *weak diameter* of a subgraph H of G is the minimum d such that any two vertices in H has distance at most d in G.

A k-(vertex-)coloring of G has *weak diameter* at most N if every monochromatic component has weak diameter at most N.

For a positive integer ℓ, G^ℓ is the graph with $V(G^\ell) = V(G)$, and $xy \in E(G^\ell)$ if and only if the distance between x and y in G is at most ℓ.

The *asymptotic dimension* of a class $\mathcal{F}$ of graphs is the minimum d such that there exists a function f such that for every $G \in \mathcal{F}$ and positive integer ℓ, G^ℓ has a $(d+1)$-coloring with weak diameter at most $f(\ell)$.

- **(Gromov):** The class of k-dimensional grids has asymptotic dimension k.
- **(Bell, Dranishnikov):** The class of trees has asymptotic dimension 1.

Open problem

Question: (Bonamy, Bousquet, Esperet, Groenland, Pirot, Scott) Does every class of graphs with polynomial expansion have finite asymptotic dimension?

- A class of graphs has *expansion at most f* if any graph obtained by contracting connected subgraphs of radius at most r in a graph in this class has average degree at most $f(r)$,
- **(Mader):** Every proper minor-closed family has constant expansion.
- **(L.):** Every proper minor-closed family has asymptotic dimension at most 2.
- **(Hume):** Any infinite class of expanders of bounded max degree has infinite asymptotic dimension.
- **(Bonamy, Bousquet, Esperet, Groenland, Pirot, Scott):** Any class of graphs with polynomial growth has finite asymptotic dimension.
- A class of graphs has *growth at most f* if for any r and any vertex in a graph in this class has at most $f(r)$ vertices with distance at most r away from it.