Matroids on graphs from rigidity theory and matrix completion: open problems

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Matroids from algebraic geometry

Definition (Algebraic matroid)

Let E be a finite set and let $V \subseteq \mathbb{C}^E$ be an irreducible variety. Let $S \subseteq E$. Denote by $\pi_S : \mathbb{C}^E \to \mathbb{C}^S$ the corresponding projection map. Then S is:

1. *independent* if $\pi_S(V) \subseteq \mathbb{C}^S$ satisfies no nontrivial polynomials
2. *a circuit* if $\overline{\pi_S(V)} \subseteq \mathbb{C}^S$ is a hypersurface and each $S' \subsetneq S$ is independent
3. *spanning* if $\dim(\pi_S(V)) = \dim(V)$

This defines the *(algebraic) matroid of* V, denoted $\mathcal{M}(V)$.

Let $E = \{1,2,3\} \times \{1,2,3\}$ and $V \subseteq \mathbb{C}^E$ be the set of 3×3 matrices of rank ≤ 1. Define $S := \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,3)\}$.

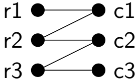
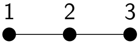
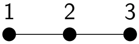
$$\pi_S(V) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & \cdot \\ \cdot & \cdot & x_{33} \end{pmatrix} : x_{11}x_{22} - x_{21}x_{22} = 0 \right\}$$

Algebraic matroid characterization problems

Problem

Let E be a finite set and $V \subseteq \mathbb{C}^E$ be an irreducible variety. Describe combinatorially the subsets $S \subseteq E$ that are independent in the algebraic matroid underlying V.

E will be (edges of) a complete (bipartite/gain) graph, so each $S \subseteq E$ can be thought of as a (bipartite/gain) graph

$\mathrm{Mat}_r^{m \times n}$	$m \times n$ matrices of rank $\leq r$	$\begin{pmatrix} 5 & \cdot & \cdot \\ -4 & -2 & \cdot \\ \cdot & 8 & 3 \end{pmatrix}$	
CM_d^n	pairwise distances among n points in $\mathbb{R}^d$	$\begin{pmatrix} 0 & 2 & \cdot \\ 2 & 0 & 3 \\ \cdot & 3 & 0 \end{pmatrix}$	
$\mathrm{Skew}_r^{n \times n}$	$n \times n$ skew-symmetric matrices of rank $\leq r$	$\begin{pmatrix} 0 & 2 & \cdot \\ -2 & 0 & 1 \\ \cdot & -1 & 0 \end{pmatrix}$	

Some characterizations

Proposition (Folklore)

$\mathcal{M}(\mathrm{Mat}_1^{m\times n})$ is the graphic matroid of $K_{m,n}$ and $\mathcal{M}(\mathrm{CM}_1^n)$ is the graphic matroid of K_n.

Theorem (Geiringer 1927, "Laman's Theorem")

A graph G is independent in $\mathcal{M}(\mathrm{CM}_2^n)$ if and only if every subgraph of G on v vertices has at most $2v-3$ edges.

Theorem (B-, 2016)

A graph $G=([n],E)$ is independent in $\mathcal{M}(\mathrm{Skew}_2^{n\times n})$ iff there exists an acyclic orientation of G that has no alternating closed trail.

Theorem (B-, 2016)

A graph $G=([m],[n],E)$ is independent in $\mathcal{M}(\mathrm{Mat}_2^{m\times n})$ iff there exists an acyclic orientation of G that has no alternating cycle.

Open problems and conjectures

Question

Can independence in $\mathcal{M}(\text{Skew}_2^{n\times n})$ or $\mathcal{M}(\text{Mat}_2^{m\times n})$ be determined in polynomial time, or is this decision problem NP-complete?

Conjecture

$\mathcal{M}(\text{Skew}_2^{n\times n})$ *is the maximum matroid (in the weak order) on* K_n *such that every copy of* K_4 *and* $K_{3,3}$ *is a circuit.*

Conjecture

$\mathcal{M}(\text{Mat}_2^{m\times n})$ *is the maximum matroid (in the weak order) on* $K_{m,n}$ *such that every copy of* $K_{3,3}$ *is a circuit. More generally,* $\mathcal{M}(\text{Mat}_r^{m\times n})$ *is the maximum matroid on* $K_{m,n}$ *such that every copy of* $K_{r+1,r+1}$ *is a circuit.*

Problem

Develop a general theory of (representable) matroids with graph symmetry.

References

The basics:

- *The algebraic combinatorial approach to matrix completion*. Király, Theran, and Tomoka 2014. https://arxiv.org/abs/1211.4116
- *Completion of tree metrics and rank-two matrices*. Bernstein 2017. https://arxiv.org/abs/1612.06797

Maximality in the weak order:

- *Abstract 3-Rigidity and Bivariate C_2^1-Splines II: Combinatorial Characterization*. Clinch, Jackson, and Tanigawa 2019. https://arxiv.org/abs/1911.00207

If you like gain graphs:

- *Generic symmetry-forced infinitesimal rigidity: translations and rotations*. Bernstein 2020. https://arxiv.org/abs/2003.10529
- Matroids of Gain Graphs in Applied Discrete Geometry. Tanigawa 2012. https://arxiv.org/abs/1207.3601