Some approaches to characterising representable matroids

Dillon Mayhew

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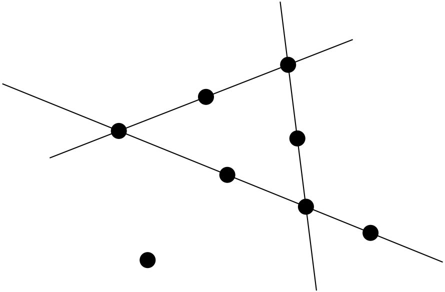
Introduction to matroids

Matroids abstract various notions of (in)dependence.



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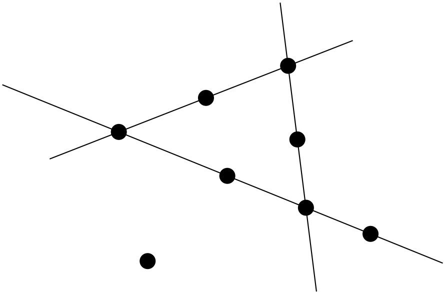

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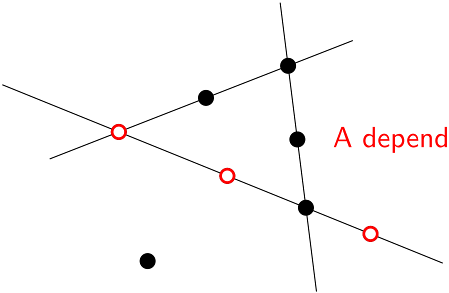

A dependent set

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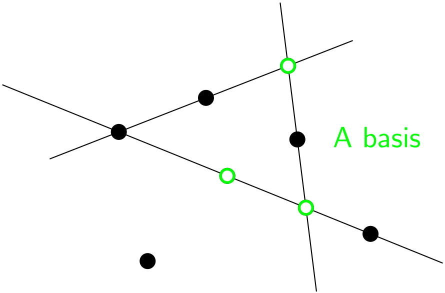

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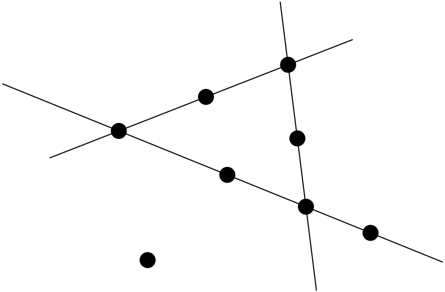

If B_1 and B_2 are bases, and $x \in B_1 - B_2$, then there is an element $y \in B_2 - B_1$ such that $(B_1 - x) \cup y$ is a basis.

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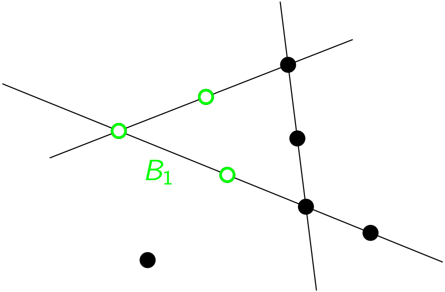

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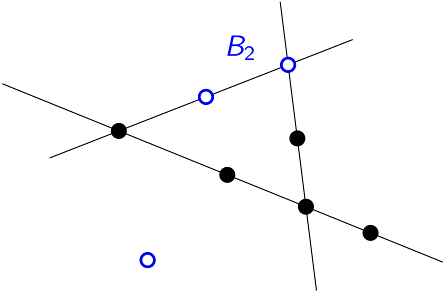

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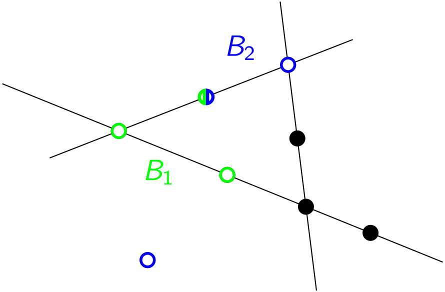

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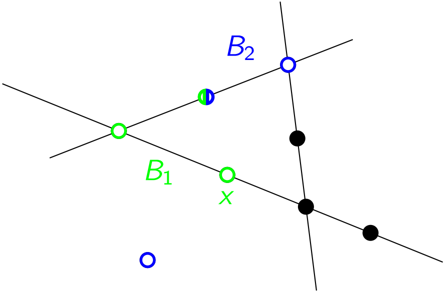

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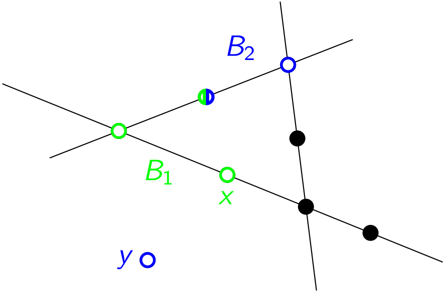

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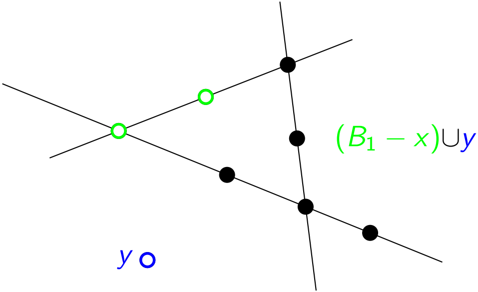

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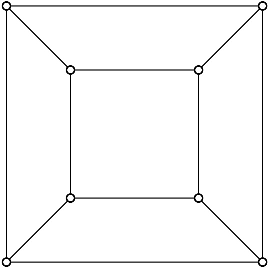

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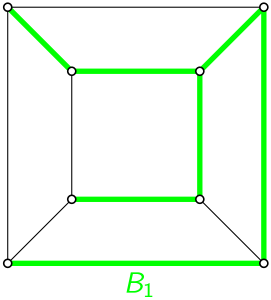

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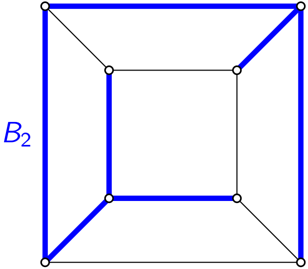

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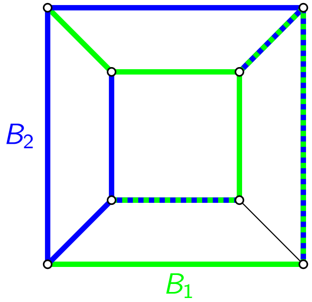

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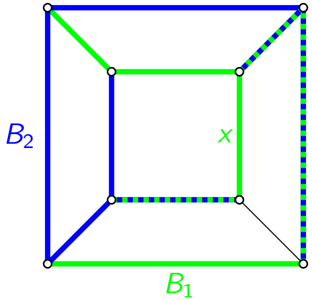

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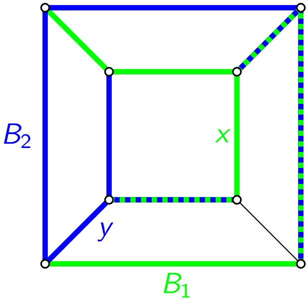

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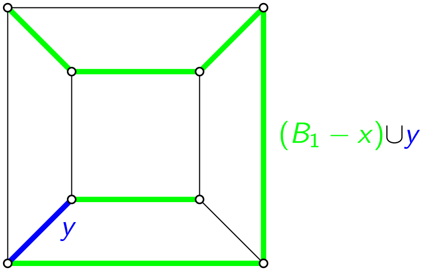

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Introduction to matroids

Definition

A matroid $M = (E, \mathcal{B})$ consists of a finite set, E, and a non-empty family, $\mathcal{B}$, of subsets of E, satisfying:

- if $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

E is called the ground set. Members of $\mathcal{B}$ are called bases.

Exercise

Bases have the same size.

Representable matroids

Matroids arise from linear (in)dependence.

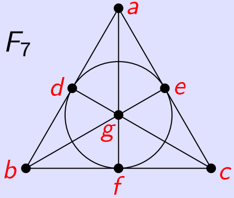
Let A be a matrix with entries from the field $\mathbb{F}$. Let E be the set of columns, and let $\mathcal{B} = \{B \subseteq E \colon B \text{ is a basis of the column-space}\}$.

Then $M[A] = (E, \mathcal{B})$ is an $\mathbb{F}$-representable matroid.

Example ($\mathbb{F} = \mathrm{GF}(2)$)

$$A = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$M[A] \cong F_7$

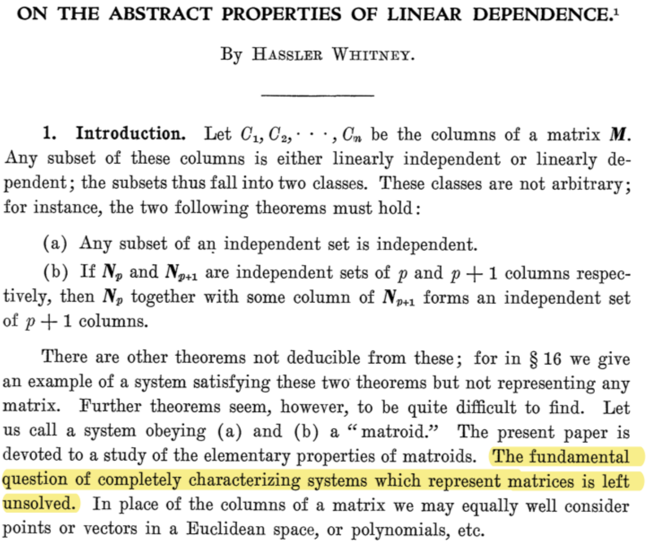


A matroid is representable if it is $\mathbb{F}$-representable for some field $\mathbb{F}$.

Whitney's problem

A significant fraction of matroid research has been inspired by a problem posed by Hassler Whitney in 1935. The problem appears on the first page of the first paper to use the word "matroid".

Whitney's problem



ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

Whitney's problem

In other words... can we characterize representable matroids?

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The classical approach to Whitney's problem has involved excluded minors.

Matroid minors

Let $M = (E, \mathcal{B})$ be a matroid, and let e be an element of E.

Definition

$M \backslash e$ is produced from M by deleting e.

$$M \backslash e = (E - e, \{B \colon B \in \mathcal{B}, e \notin \mathcal{B}\})$$

(if at least one $B \in \mathcal{B}$ does not contain e).

M/e is produced from M by contracting e.

$$M/e = (E - e, \{B - e \colon B \in \mathcal{B}, e \in \mathcal{B}\})$$

(if at least one $B \in \mathcal{B}$ contains e).

A minor of M is produced by a (possibly empty) sequence of deletions and contractions.

Excluded minors

Definition

Let $\mathcal{M}$ be a class of matroids. $\mathcal{M}$ is minor-closed if any minor of any matroid in $\mathcal{M}$ is also in $\mathcal{M}$.

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Let $\mathbb{F}$ be a field. The class of $\mathbb{F}$-representable matroids is closed under minors.

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Definition

Let $\mathcal{M}$ be a minor-closed class of matroids. The matroid $M = (E, \mathcal{B})$ is an excluded minor for $\mathcal{M}$ if $M \notin \mathcal{M}$, but $M \backslash e \in \mathcal{M}$ and $M/e \in \mathcal{M}$ for all $e \in E$.

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Exercise

Let $\mathcal{M}$ be a minor-closed class. A matroid is contained in $\mathcal{M}$ if and only if it does not contain an excluded minor (as a minor).

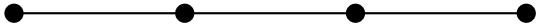
Excluded minors

These results mean that we can characterize the class of $\mathbb{F}$-representable matroids by giving a list of excluded minors.

Existing excluded-minor theorems

Theorem (Tutte — 1958)

The only excluded minor for the class of GF(2)-representable matroids is $U_{2,4}$.



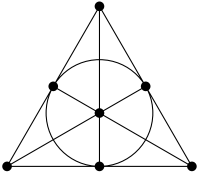
$U_{2,4}$

Existing excluded-minor theorems

Theorem (Bixby / Seymour — 1979)

The excluded minors for the class of GF(3)-representable matroids are $U_{2,5}$, $U_{3,5}$, F_7, and F_7^*.

$U_{2,5}$



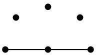
F_7

Existing excluded-minor theorems

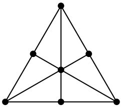
Theorem (Geelen, Gerards, Kapoor — 2000)

The excluded minors for the class of GF(4)-representable matroids are $U_{2,6}$, $U_{4,6}$, P_6, F_7^-, $(F_7^-)^*$, P_8, and P_8''.

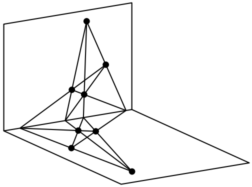
$U_{2,6}$



P_6



F_7^-



P_8

Existing excluded-minor theorems

Theorem (Tutte — 1958)

The excluded minors for GF(2)- and GF(3)-representable matroids are $U_{2,4}$, F_7, and F_7^*.

Theorem (Geelen, Gerards, Kapoor — 2000)

The excluded minors for GF(3)- and GF(4)-representable matroids are $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, F_7^-, $(F_7^-)^*$, and P_8.

Theorem (Geelen / Hall, Mayhew, Van Zwam — 2010)

The excluded minors for GF(3)-, GF(4)-, and GF(5)-representable matroids are $U_{2,5}$, $U_{3,5}$, F_7, $(F_7)^*$, F_7^-, $(F_7^-)^*$, P_8, $\mathrm{AG}(2,3)\backslash e$, $(\mathrm{AG}(2,3)\backslash e)^*$, and $\Delta_T(\mathrm{AG}(2,3)\backslash e)$.

Future excluded-minor theorems?

Ongoing projects (Mayhew, Whittle, Van Zwam, and others).

Project

Find the excluded minors for the class of $\mathbb{H}_5$-representable matroids.

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564 found by Royle, 4 found by Kingan, 2 found by Pendavingh.

Future excluded-minor theorems?

Conjecture

The excluded minors for GF(3)- and GF(5)-representable matroids are $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, $\mathrm{AG}(2,3)\backslash e$, $(\mathrm{AG}(2,3)\backslash e)^*$, $\Delta_T(\mathrm{AG}(2,3)\backslash e)$, T_8, N_1, and N_2.

This is a necessary step towards finding the excluded minors for GF(5)-representability.

Rota's conjecture

The most famous problem in matroid theory...

Rota's conjecture (1971)

If $\mathbb{F}$ is a finite field, then there are finitely many excluded minors for $\mathbb{F}$-representability.

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Rota's conjecture (1971)

If $\mathbb{F}$ is a finite field, then there are finitely many excluded minors for $\mathbb{F}$-representability.

... has just been solved.

Theorem (Geelen, Gerards, Whittle — announced 2013)

Rota's conjecture holds.

Whitney's problem

The classical approach to Whitney's problem has involved excluded minors.

Another approach uses model theory — the theory of formal languages.

Matroid axioms

Let $M = (E, \mathcal{B})$ be a matroid.

A subset $X \subseteq E$ is independent in M if $X \subseteq B$ for some $B \in \mathcal{B}$.

Let E be a finite set, let $\mathcal{I} \subseteq 2^E$ be a collection of subsets. Then $\mathcal{I}$ is the collection of independent sets of a matroid if and only if:

I1. $\mathcal{I} \neq \emptyset$

I2. $I \in \mathcal{I}$ and $I' \subseteq I$ implies $I' \in \mathcal{I}$

I3. If I and I' are maximal subsets in $\mathcal{I}$, and $x \in I - I'$, then there exists $y \in I' - I$ such that $(I - x) \cup y$ is a maximal subset in $\mathcal{I}$.

Can we add finitely many sentences to this list of axioms in such a way that we characterize representable matroids (using the same logical language)?

Vámos's Theorem

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Representable V-matroids cannot be characterized by adding a single sentence in V-logic to the (infinite) list of V-matroid axioms.

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Because of more recent progress, we now plan on publishing 'Yes, the missing axiom of matroid theory is lost forever'.

Representability in MS_0

Theorem (Mayhew, Newman, Whittle — 2013)

There is no finite sentence, S, in MS_0 such that $\{\mathbf{I1}, \mathbf{I2}, \mathbf{I3}, S\}$ exactly characterizes the set of representable matroids.

Monadic second-order logic

Features of MS_0

- Subset variables $X_1, X_2, X_3, \ldots$
- A unary independence predicate, Ind
- A unary singleton predicate, Sing
- Relations, $\subseteq$, $=$
- Logical symbols, $\exists$, $\forall$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$

We can axiomatize matroids in MS_0, and for any fixed matroid N, we can make the statement 'contains a minor isomorphic to N'.

Preliminaries

Definition

Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids such that $E_1 \cap E_2 = \emptyset$. Then

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 \colon B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\})$$

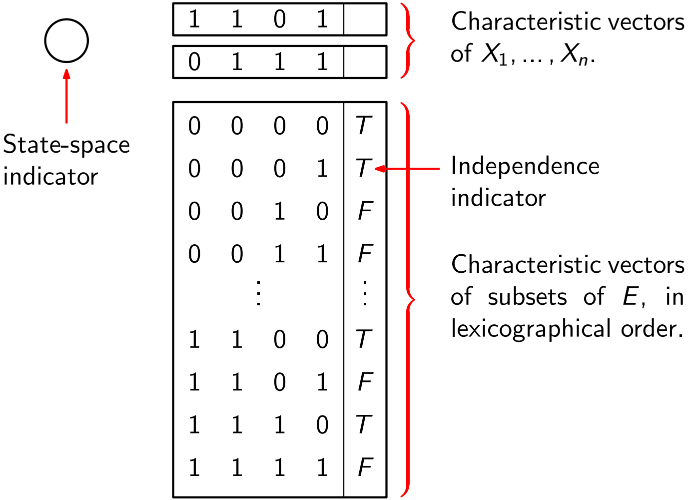
is a matroid, the direct sum of M_1 and M_2.

Proposition

For any prime p, there is a matroid, $\mathrm{PG}(2, p)$, the projective plane of order p, that is representable only over fields of characteristic p.

Finite-state automata for matroids

We construct a finite-state machine that operates on the matroid $M = (E, \mathcal{I})$ and subsets $X_1, \dots, X_n \subseteq E$.



State-space indicator

1	1	0	1	
0	1	1	1	

Characteristic vectors of $X_1, \dots, X_n$.

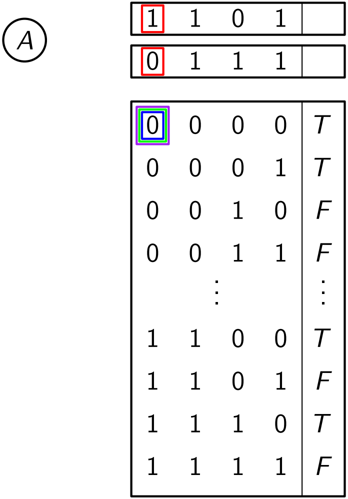
0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

Independence indicator

Characteristic vectors of subsets of E, in lexicographical order.

Finite-state automata for matroids

The machine employs readers to crawl through the matrix.



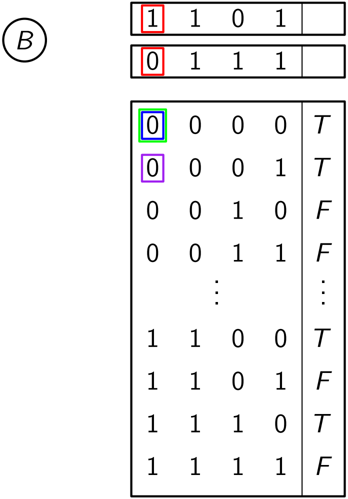
1	1	0	1	

0	1	1	1	

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

Finite-state automata for matroids

Depending on its current state, and the symbols under the readers, the machine can direct the readers to move down one space, or wait.



1	1	0	1	

0	1	1	1	

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

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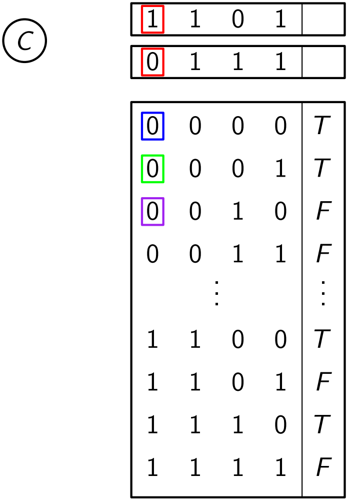

1	1	0	1	

0	1	1	1	

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

Finite-state automata for matroids

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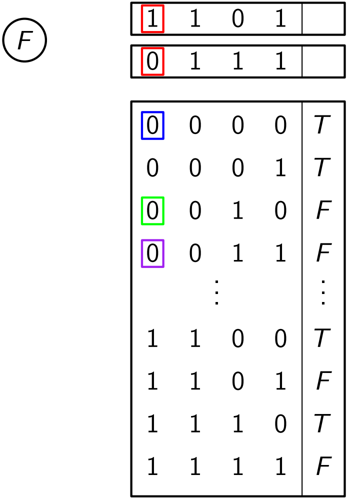


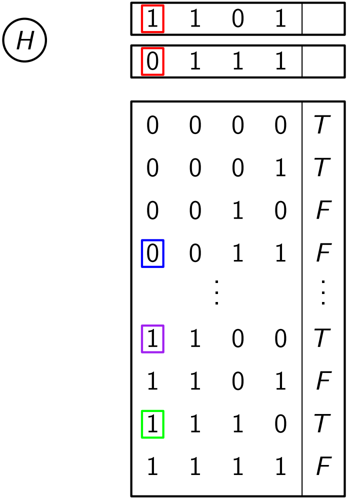
1	1	0	1	

0	1	1	1	

0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

Finite-state automata for matroids

When all the readers have reached a waiting state...

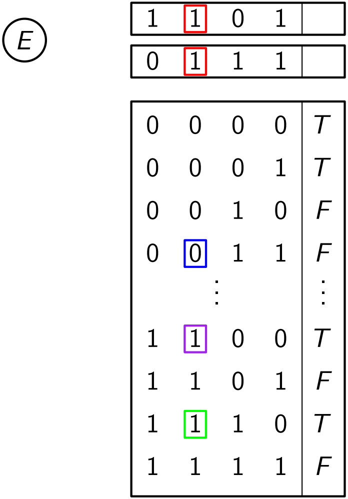


1	1	0	1	
0	1	1	1	

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

Finite-state automata for matroids

... they all move one space right, and the process continues.



1	1	0	1	

0	1	1	1	

0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

Finite-state automata for matroids

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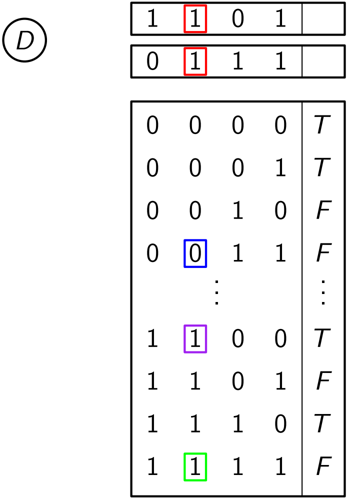


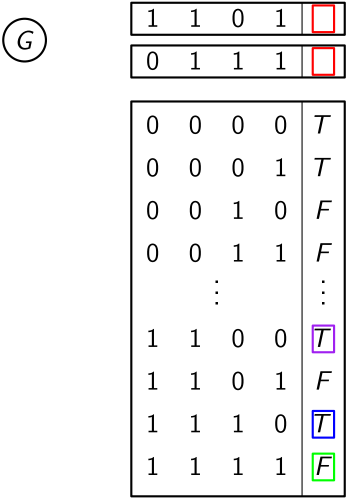
1	1	0	1	

0	1	1	1	

0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

Finite-state automata for matroids

The machine halts when all readers are in the final column. It accepts $(M; X_1, \dots, X_n)$ if it is in an accepting state when it halts.



1	1	0	1	□

0	1	1	1	□

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	$\boxed{T}$
1	1	0	1	F
1	1	1	0	$\boxed{T}$
1	1	1	1	$\boxed{F}$

Finite-state automata for matroids

We also want the machine to operate on pairs of matrices, which represent direct sums of matroids.

1	1	0	1	

0	1	1	1	

0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

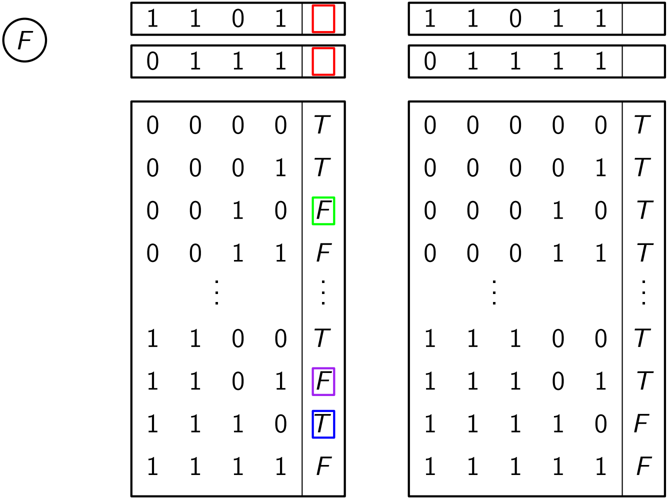
1	1	0	1	1	

0	1	1	1	1	

0	0	0	0	0	*T*
0	0	0	0	1	*T*
0	0	0	1	0	*T*
0	0	0	1	1	*T*
	⋮				⋮
1	1	1	0	0	*T*
1	1	1	0	1	*T*
1	1	1	1	0	*F*
1	1	1	1	1	*F*

Finite-state automata for matroids

When the readers reach the final column of one matrix...



1	1	0	1	☐
0	1	1	1	☐

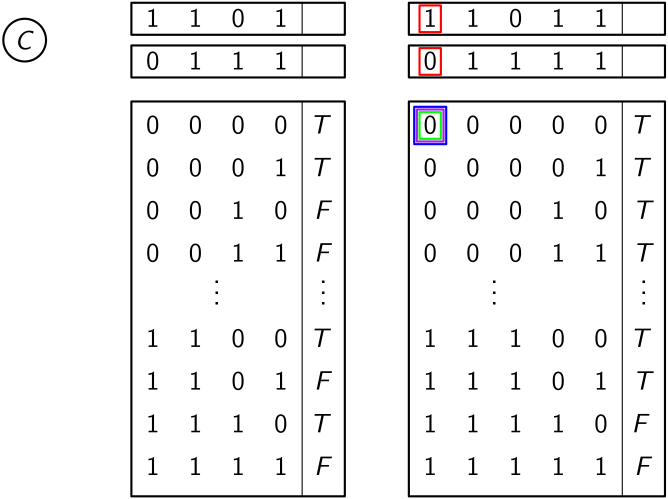
0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

1	1	0	1	1	
0	1	1	1	1	

0	0	0	0	0	*T*
0	0	0	0	1	*T*
0	0	0	1	0	*T*
0	0	0	1	1	*T*
		⋮			⋮
1	1	1	0	0	*T*
1	1	1	0	1	*T*
1	1	1	1	0	*F*
1	1	1	1	1	*F*

Finite-state automata for matroids

When the readers reach the final column of one matrix... they start in the top left corner of the next.



1	1	0	1	
0	1	1	1	

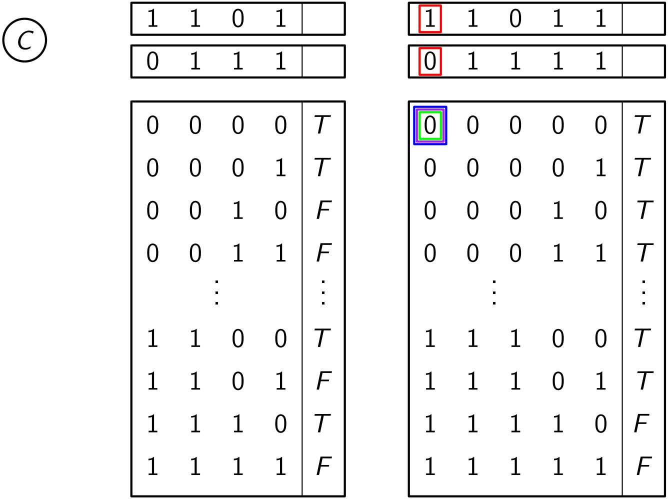
0	0	0	0	*T*
0	0	0	1	*T*
0	0	1	0	*F*
0	0	1	1	*F*
	⋮			⋮
1	1	0	0	*T*
1	1	0	1	*F*
1	1	1	0	*T*
1	1	1	1	*F*

1	1	0	1	1	
0	1	1	1	1	

0	0	0	0	0	*T*
0	0	0	0	1	*T*
0	0	0	1	0	*T*
0	0	0	1	1	*T*
	⋮				⋮
1	1	1	0	0	*T*
1	1	1	0	1	*T*
1	1	1	1	0	*F*
1	1	1	1	1	*F*

Finite-state automata for matroids

For any statement ψ in MS_0, there is a machine which will accept $(M; X_1, \ldots, X_n)$ if and only if ψ applied to $(M; X_1, \ldots, X_n)$ is true.



1	1	0	1	
0	1	1	1	

0	0	0	0	T
0	0	0	1	T
0	0	1	0	F
0	0	1	1	F
	⋮			⋮
1	1	0	0	T
1	1	0	1	F
1	1	1	0	T
1	1	1	1	F

1	1	0	1	1	
0	1	1	1	1	

0	0	0	0	0	T
0	0	0	0	1	T
0	0	0	1	0	T
0	0	0	1	1	T
	⋮				⋮
1	1	1	0	0	T
1	1	1	0	1	T
1	1	1	1	0	F
1	1	1	1	1	F

Theorem proof

Theorem (Mayhew, Newman, Whittle — 2013)

There is no finite sentence, S, in MS_0 such that $\{\mathbf{I1}, \mathbf{I2}, \mathbf{I3}, S\}$ exactly characterizes the set of representable matroids.

Theorem proof

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Assume such a sentence exists. Consider the machine that accepts M if and only if S holds for M. This machine has finitely many states.

Theorem proof

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Assume such a sentence exists. Consider the machine that accepts M if and only if S holds for M. This machine has finitely many states.

There exist distinct primes p and p' such that the machine halts in the same accepting state when applied to $\mathrm{PG}(2, p)$ and $\mathrm{PG}(2, p')$.

Theorem proof

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There is no finite sentence, S, in MS_0 such that $\{\mathbf{I1}, \mathbf{I2}, \mathbf{I3}, S\}$ exactly characterizes the set of representable matroids.

Assume such a sentence exists. Consider the machine that accepts M if and only if S holds for M. This machine has finitely many states.

There exist distinct primes p and p' such that the machine halts in the same accepting state when applied to $\mathrm{PG}(2, p)$ and $\mathrm{PG}(2, p')$.

What state does the machine halt in when applied to $\mathrm{PG}(2, p) \oplus \mathrm{PG}(2, p)$ and $\mathrm{PG}(2, p') \oplus \mathrm{PG}(2, p)$?