Matroids and MySQL — What to do with big data sets?

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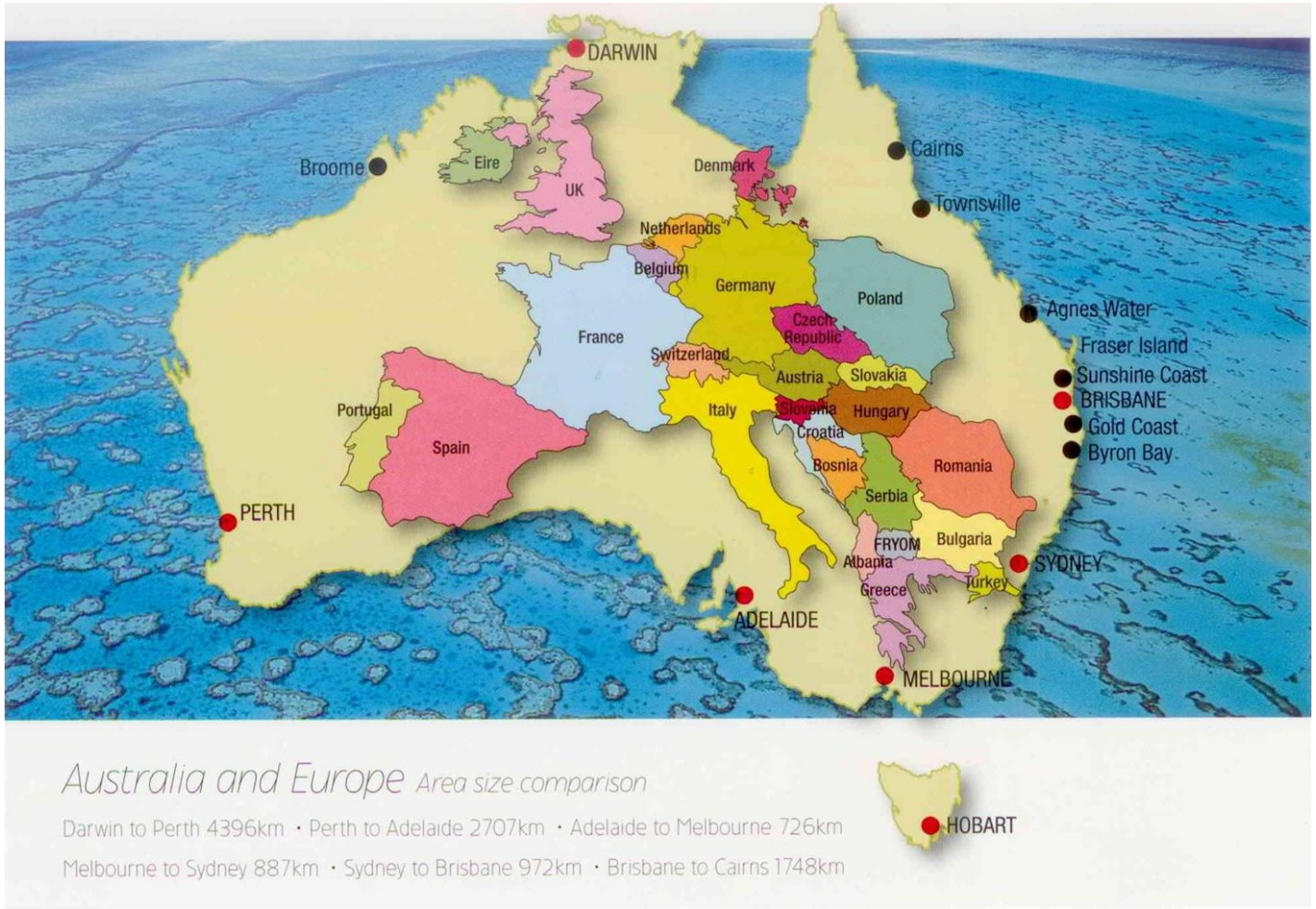
DARWIN
Broome
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Switzerland
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Spain
Bosnia
Romania
Serbia
PERTH
FRYOM
Bulgaria
Albania
SYDNEY
Greece
Turkey
ADELAIDE
MELBOURNE
Australia and Europe Area size comparison
Darwin to Perth 4396km · Perth to Adelaide 2707km · Adelaide to Melbourne 726km
Melbourne to Sydney 887km · Sydney to Brisbane 972km · Brisbane to Cairns 1748km
HOBART

Perth

37TH AUSTRALASIAN CONFERENCE ON COMBINATORIAL MATHEMATICS AND COMBINATORIAL COMPUTING

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Combinatorial Catalogues

There is a long history of combinatorial mathematicians constructing *lists*, *catalogues* or *census* [sic] of combinatorial objects, such as groups, graphs, designs and geometries.

Some of this work *substantially* predates electronic computers:

- GA Miller's work in the late 1800s on "*substitution groups*"
- The 1930s "Foster Census" of *cubic symmetric graphs*

but it really exploded from the 1960s onwards.

A catalogue of combinatorial geometries

Author(s): John E. Blackburn; Henry H. Crapo; Denis A. Higgs.
Journal: Math. Comp. **27** (1973), 155-166.
MSC: Primary 05B35

Using Catalogues

Although catalogues can only contain information about the *very smallest objects* of any given class, this is often useful.

In addition to the examples and/or counterexamples directly contained in a catalogue, "*small case information*" can be crucial to the development and proof of theoretical results.

- Small cases can identify *interesting cases* for a problem

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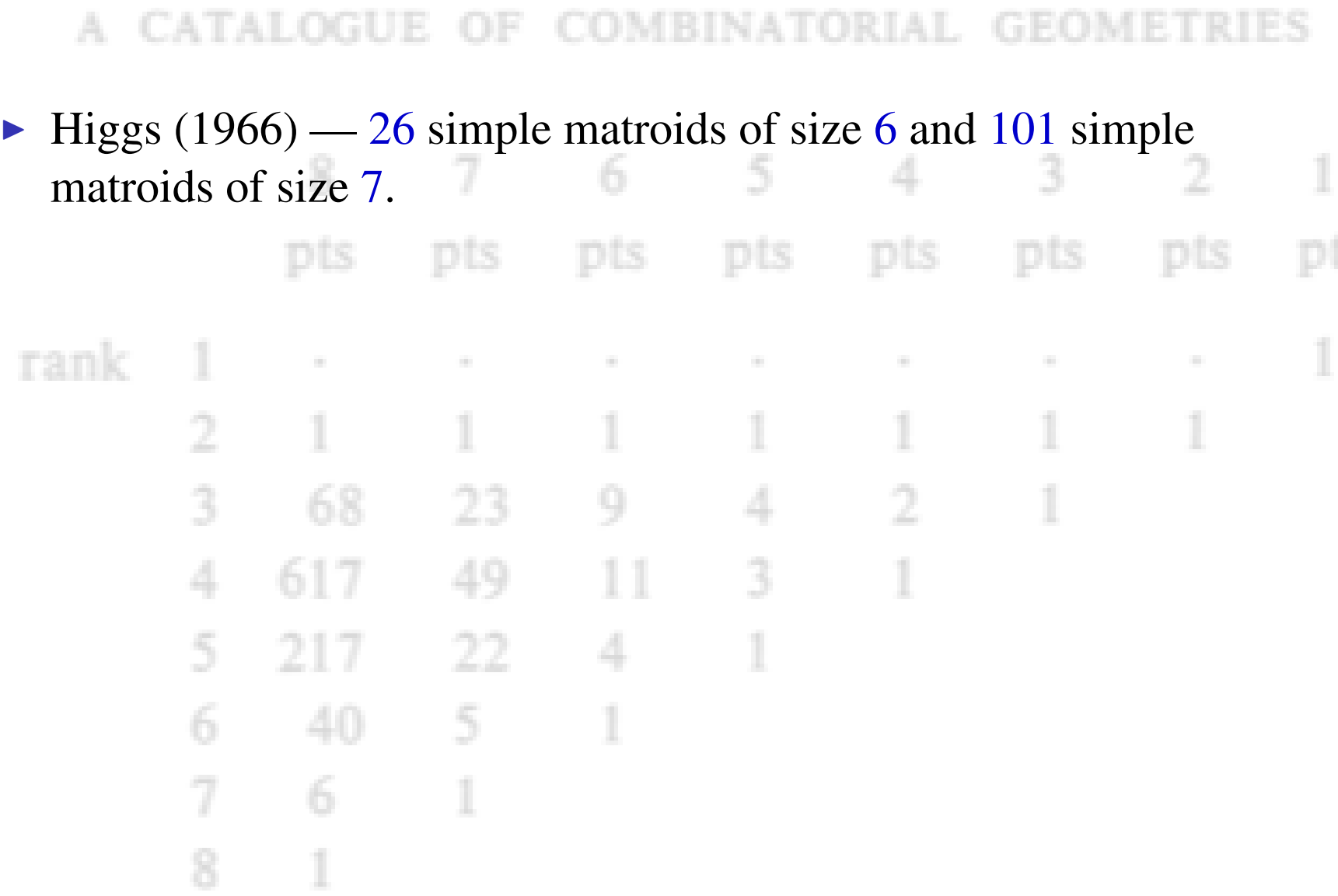
- Small cases can identify *interesting cases* for a problem
- Small cases can *reveal* the emergence of patterns
- Small cases can *obscure* the emergence of patterns

General Matroids

- Higgs (1966) — 26 simple matroids of size 6 and 101 simple matroids of size 7.



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- Mayhew & Royle (2009) — 376467 matroids of size 9.
- Östergård (2010) — 2630337889305 sparse paving matroids of size 10 and rank 5

At least for the next few years, doing anything other than *counting* the 10-element matroids is well out of reach.

By comparison, *group theorists* have lists of all groups up to about order 2000.

Representable Matroids

The classes of *binary* or *ternary* matroids etc. are much smaller, and so catalogues can pushed further.

- Catalogues of *all* matroids of small enough size, rank
- Catalogues of *minor-closed classes* for slightly larger sizes

Then the question arises:

- What is the best way to *store*, *use* and *provide access* to these catalogues?

Binary and Ternary Matroids

For binary and ternary matroids, additional useful properties hold:

- There is a 1-1 correspondence between *simple matroids* of rank at most d and *subsets of points* of the projective space $PG(d-1,q)$.
- Two matroids are *isomorphic* if and only if the corresponding subsets of points are *equivalent* under the action of the group $PGL(d,q)$ of *linear transformations* of the projective space.

Therefore, if we let Γ be the *point-hyperplane incidence graph* of $PG(d-1,q)$, one technique is to use an *orderly algorithm* to construct subsets of the "*point-vertices*" of Γ that are inequivalent under $\mathrm{Aut}(\Gamma)$.

An orderly algorithm

An *orderly algorithm* is a construction algorithm that generates objects *uniquely up to isomorphism*.

Assume that $\mathcal{L}_k$ contains one representative from each orbit on k-subsets of point-type vertices of Γ.

For each $X \in \mathcal{L}_k$ do the following:

- Compute orbits of $\mathrm{Aut}(\Gamma)_X$
- For each point-type orbit representative x
  - Form $X' = X \cup \{x\}$
  - Compute canonically labelled graph with X' distinguished
  - If x is in the same orbit as the point with lowest canonical label, then accept X' else reject

Then all the accepted $(k+1)$-subsets form $\mathcal{L}_{k+1}$.

Why does this work?

This works because of two key facts:

- The recipe "canonically label and remove lowest numbered element" defines a unique (up to isomorphism) *deconstruction path* for any binary matroid.
  - The *accept/reject* rule for the orderly algorithm ensures that the construction goes — *in reverse* — along this path.
- Adding just one orbit representative guarantees that every matroid is generated only once from its successor on the path.

Numbers of binary matroids

	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$e=7$	1	5	8	5	1	
$e=8$		6	15	14	6	1
$e=9$		5	29	38	22	7
$e=10$		4	46	105	80	32
$e=11$		3	64	273	312	151
$e=12$		2	89	700	1285	821
$e=13$		1	112	1794	5632	5098
$e=14$		1	128	4579	26792	37191
$e=15$		1	144	11635	137493	320663
$e=16$			145	29091	745413	3186083
$e=17$			129	70600	4145064	34799393

Binary matroids of rank at most 6

0	1	16	29236	32	46875728	48	11780
1	1	17	70729	33	44065939	49	4708
2	1	18	164818	34	38939167	50	1907
3	2	19	366180	35	32339616	51	791
4	3	20	770299	36	25238329	52	340
5	5	21	1528238	37	18504126	53	155
6	10	22	2852575	38	12742321	54	72
7	19	23	5002828	39	8239590	55	35
8	35	24	8239590	40	5002828	56	19
9	72	25	12742321	41	2852575	57	10
10	155	26	18504126	42	1528238	58	5
11	340	27	25238329	43	770299	59	3
12	791	28	32339616	44	366180	60	2
13	1907	29	38939167	45	164818	61	1
14	4708	30	44065939	46	70729	62	1
15	11780	31	46875728	47	29236	63	1

Counting Binary or Ternary Matroids

We can be confident in the results because the *exact numbers* of simple binary or ternary matroids of any given rank can be determined by *Pólya's Enumeration Theorem*

- Determine the *cycle index* $Z(PGL(d,q))$ of $PGL(d,q)$: The cycle index is the multivariate polynomial in variables $\{X_1, X_2, \ldots\}$ given by

$$Z(G; X_1, X_2, \ldots) := \frac{1}{|G|} \sum_{g \in G} \prod X_k^{n_k(g)}.$$

  where $n_k(g)$ is the number of cycles of length k in g.
- Substitute $1 + x^i$ for each variable X_i.
- The coefficient of x^j is then the number of isomorphism classes of matroids of size j and rank at most d.

Example

In Magma

```
> g := PGammaL(4,2);
> cp := CycleIndexPolynomial(g);
> cp;
1/20160*x[1]^15 + 1/192*x[1]^7*x[2]^4 + 1/96*x[1]^3*x[2]^6 +
    1/16*x[1]^3*x[2]^2*x[4]^2 + 1/18*x[1]^3*x[3]^4 +
    1/6*x[1]*x[2]*x[3]^2*x[6] + 1/8*x[1]*x[2]*x[4]^3 +
    2/7*x[1]*x[7]^2 + 1/180*x[3]^5 + 1/12*x[3]*x[6]^2 +
    1/15*x[5]^3 + 2/15*x[15]
```

Note: For some reason, the `CycleIndexPolynomial` command does not appear in the documentation for Magma. I stumbled on it because it was the obvious name to use.

Rank 4 binary matroids

Substituting $1 + x^i$ for X_i yields the *generating function*

$$x^{15} + x^{14} + x^{13} + 2x^{12} + 3x^{11} + 4x^{10} + 5x^9 + 6x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 + x + 1$$

and reading off the coefficients we see that there are *3* binary matroids of size 11 and rank at most 4.

CAN WE PUSH FURTHER?

The exact numbers give us a good idea of where to stop computing!

	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$e = 7$	4	19	15	5	1
$e = 8$	3	44	61	26	6
$e = 9$	3	91	277	162	40
$e = 10$	2	199	1439	1381	375
$e = 11$	1	401	8858	17200	5923
$e = 12$	1	806	62311	311580	182059
$e = 13$	1	1504	459828	6876068	9427034
$e = 14$	0	2659	3346151	159373844	608045192
$e = 15$	0	4304	23246482	3609085016	40932394177

Numbers of ternary matroids

SQL

I decided to try storing the data in an actual *relational database* so that it can be queried using *SQL* (Structured Query Language).

Advantages

- Single fixed location for data that is (potentially) accessible from multiple computers
- Relatively easy to integrate web-based access for other users, or even direct programmatic access
- Properties can be added without breaking existing code

Disadvantages

- Rigid format can be limiting
- SQL can be awkward, unnatural, opaque and inefficient

MINIMAL SETUP

At a minimum we want to *store* the matroids and some basic data about them. Thus we create a *database table* called `rank6`.

```
CREATE TABLE binarymatroids (
  id INT PRIMARY KEY,
  size INT,
  rank INT,
  points TEXT);
```

and add the data

```
INSERT INTO binarymatroids VALUES(1,1,1,"1");
INSERT INTO binarymatroids VALUES(2,2,2,"1 2");
INSERT INTO binarymatroids VALUES(3,3,2,"1 2 3");
...
...
```

Minors

To make the database more than mildly useful, it is essential to capture the *minor relationship between* matroids, rather than just properties of single matroids.

It is infeasible to list all the minors for each matroid directly, but the results of *single-element* deletions and contractions can be stored.

```
CREATE TABLE binaryMinors (
  id INT,
  point INT,
  delid INT,
  conid INT);
```

```
CREATE TABLE binaryMinors2 (
  id INT,
  minor INT);
```

To populate this table we need to *find* and *identify* all the single-element deletions and contractions of every matroid.

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Identifying a matroid

Each matroid M is assigned a unique *hash code* computed from the graph Γ as follows:

- Use `nauty` to compute the *canonical labelling* of Γ with the points of M distinguished (i.e. "coloured").
- Use a hashing algorithm (in this case SHA) to give each matroid a *fingerprint*.
- Store all fingerprints in lexicographic (alphabetical) order.

```
00000012ac0841740437753a6fa923d41e0e4152 33932540
000000dd31abd8c7f2ddd03d4e8ec7b985d1820b 23143985
000001c4b2413c04851d3fba51d874b04973efe1 05468409
```

An unknown matroid can quickly be identified by computing its fingerprint and then using a *binary search*.

A CANONICAL FORM

As the output of `nauty` can be machine-dependent, it would be better to store a *canonical form* for each matroid.

One candidate would be to take the *lexicographically minimal* matroid equivalent to the given one.

```
gap> LoadPackage("grape");
true
gap> g := PGL(6,2);
<permutation group of size 20158709760 with 2 generators>
gap> m := [1,4,6,9,12,20,22];
[ 1, 4, 6, 9, 12, 20, 22 ]
gap> SmallestImageSet(g,m);
[ 1, 2, 4, 7, 8, 16, 25 ]
```

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```

But in general, this is *too slow* to compute for large numbers of matroids.

COMPUTING THE MINOR RELATIONSHIP

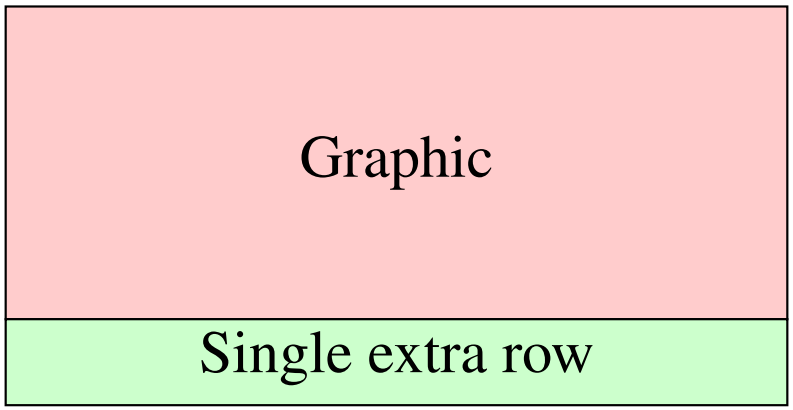
For each matroid/point pair, two minors are *constructed* and *identified* and the information stored in the `binaryMinors` table.

```
mysql> SELECT * FROM binaryMinors WHERE id = 1000;
+------+-------+-------+-------+
| id   | point | delid | conid |
+------+-------+-------+-------+
| 1000 |     1 |   417 |    77 |
| 1000 |     2 |   382 |   125 |
               ..........
               ..........
| 1000 |    26 |   384 |   251 |
| 1000 |    32 |   374 |   309 |
| 1000 |    64 |   374 |   309 |
| 1000 |   125 |   374 |   309 |
+------+-------+-------+-------+
```

This is a *huge computation* due to the sheer number of matroids but is only done once.

SAMPLE USE

An *even cycle matroid* is a binary matroid M that with a representing matrix of the form



Equivalently M is even cycle if there is binary element e such that $(M + e)/e$ is graphic.

This property is minor-closed — what are the *excluded minors* for even cycle matroids?

SETTING UP THE DATABASE

We'll add the property `isEvenCycle` to the database

```
ALTER TABLE binarymatroids
   ADD COLUMN (isEvenCycle tinyint(1));
```

Then we actually have to compute which matroids *are* even cycle.

```
def is_evencycle(m):
    return True in [n.contract('X').is_graphic()
                    for n in m.linear_extensions('X')]
```

and enter this data into the database:

```
....
UPDATE binarymatroids SET isEvenCycle = 1 WHERE id = 102323;
UPDATE binarymatroids SET isEvenCycle = 1 WHERE id = 102324;
UPDATE binarymatroids SET isEvenCycle = 0 WHERE id = 102325;
.....
```

THE FINAL COMMAND

```
SELECT B.id, B.points FROM binarymatroids B
  WHERE B.size < 17 AND B.isEvenCycle = 0
  AND NOT EXISTS
     (SELECT * FROM binarymatroids B2, binaryminors2 BM
        WHERE B.id = BM.id AND
            BM.minor = B2.id AND B2.isEvenCycle = 0);
```

In pictures

id	isEC
⋮	⋮
233	0
234	0
235	0
236	0
⋮	⋮

B

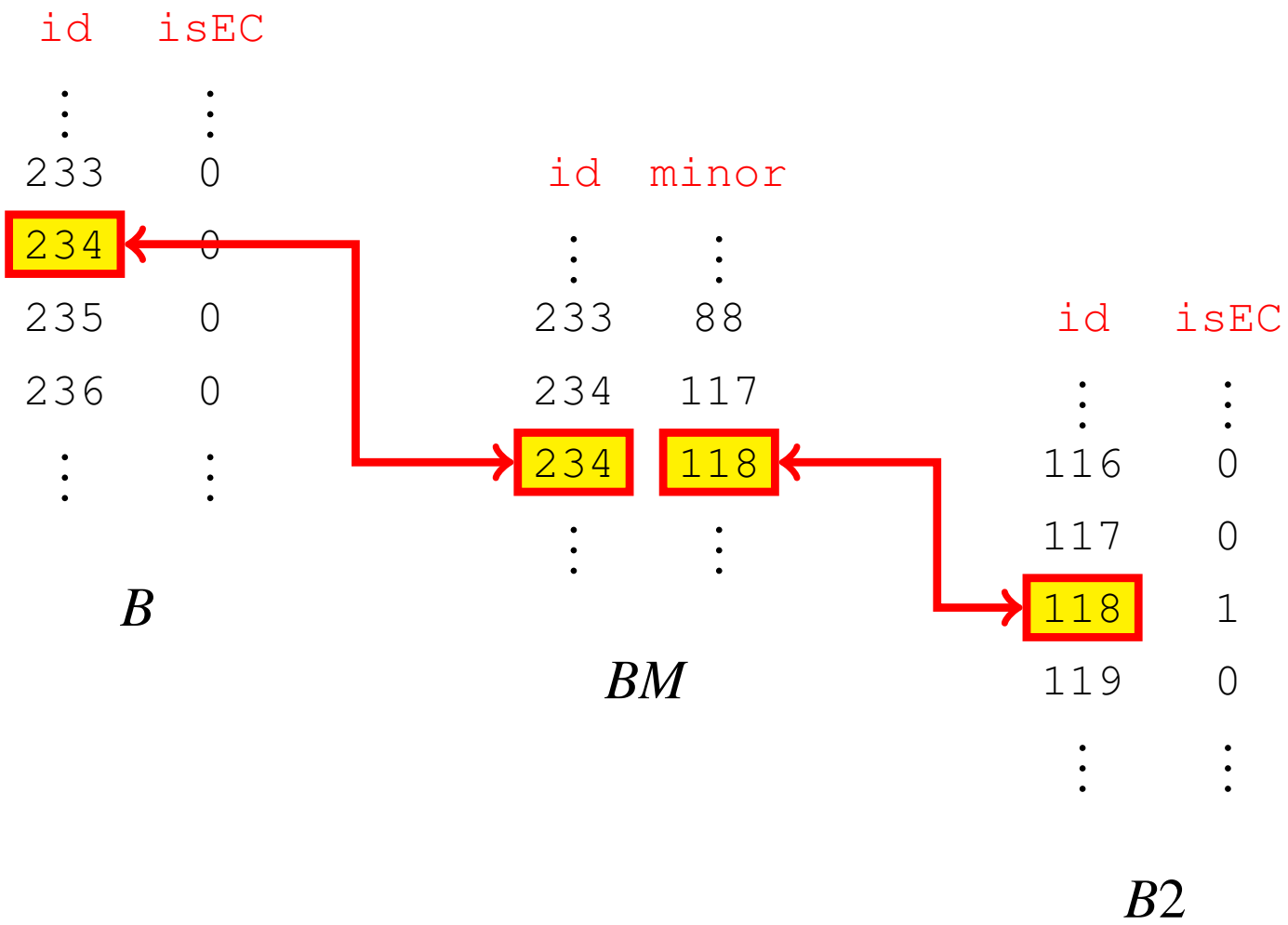
id	minor
⋮	⋮
233	88
234	117
234	118
⋮	⋮

BM

id	isEC
⋮	⋮
116	0
117	0
118	1
119	0
⋮	⋮

$B2$

In pictures


id isEC
233 0
234 0
235 0
236 0
B
id minor
233 88
234 117
234 118
BM
id isEC
116 0
117 0
118 1
119 0
B2

And the final

```
+---------+-----------------------------------------------------+
| id      | points                                              |
+---------+-----------------------------------------------------+
|     748 | 1 2 3 4 8 16 29 32 46 52 59                         |
|     774 | 1 2 4 8 15 16 21 32 41 51 63                        |
|    1343 | 1 2 3 4 5 8 9 14 16 22 26 29                        |
|    1346 | 1 2 4 7 8 11 13 14 16 21 25 31                      |...
...
...
| 4397271 | 1 2 4 8 16 18 28 32 64 66 127 128 133 143 179 208   |
| 4397878 | 1 2 4 8 16 28 32 50 64 80 127 128 133 143 179 208   |
| 4397915 | 1 2 4 8 10 16 28 32 51 64 127 128 133 143 179 208   |
| 4398085 | 1 2 4 5 8 16 28 32 64 67 127 128 133 143 179 208    |
| 4401402 | 1 2 4 8 9 16 32 64 127 128 130 133 139 143 179 208  |
| 4401801 | 1 2 4 8 9 16 32 54 64 127 128 131 133 143 179 208   |
| 4401823 | 1 2 4 8 9 16 32 64 127 128 131 133 143 178 179 208  |
| 4434811 | 1 2 4 7 8 16 32 39 64 65 127 128 143 148 179 205    |
| 4435762 | 1 2 4 7 8 16 32 64 125 127 128 143 148 156 179 205  |
| 4438645 | 1 2 4 8 12 16 24 32 64 125 127 128 143 148 179 205  |
| 4446827 | 1 2 4 8 16 32 44 64 65 127 128 143 148 179 180 205  |
| 4506185 | 1 2 4 5 8 16 32 64 127 128 131 136 141 143 179 213  |
| 4507486 | 1 2 4 8 16 32 64 66 127 128 131 143 160 179 197 213 |
+---------+-----------------------------------------------------+
717 rows in set (2 min 36.41 sec)
```

How can this be provided in Sage?

Demo

Problems

Many problems:

- Technical problems
- Non-technical problems
  Too much data, but not enough proofs!