# Counting Matroids, Entropy and Compression

Rudi Pendavingh

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**Question**

*How do you construct a random matroid?*

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'Asymptotically almost all matroids have property $\mathcal{P}$' if

$$\lim_{n\to\infty} \frac{\#\{M \in \mathbb{M}_n : M \text{ has property } \mathcal{P}\}}{\#\mathbb{M}_n} = 1$$

where $\mathbb{M}_n := \{M \text{ matroid} : E(M) = \{1, \ldots, n\}\}$.

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The following are due to Mayhew, Newman, Welsh & Whittle (2010):

**Conjecture**

*Asymptotically, almost all matroids are sparse paving.*

**Conjecture**

*For any $k$, asymptotically all matroids are $k$-connected.*

**Conjecture**

*If $N$ is a sparse paving matroid, then a.a. matroids have an $N$-minor.*

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# Part I: Counting matroids by Entropy

# The Loomis-Whitney inequality

**Theorem (Loomis & Whitney, 1949)**

*Let* $S \subseteq \mathbb{Z}^3$ *be a finite set. Then*

$$|S|^2 \leq |S_{xy}|\ |S_{xz}|\ |S_{yz}|$$

*where* $S_{uv}$ *denotes the projection of* $S$ *on the* $uv$*-plane.*

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A modern proof uses *Shearers' Entropy Lemma*

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### Theorem (Shearer, 1986)

*Suppose $X$ takes value in $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Let $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ be such that $\#\{i \mid j \in A_i\} \geq k$ for $j = 1, \ldots, n$. Then*

$$kH(X) \leq \sum_{i=1}^{m} H(X_{A_i})$$

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Proof: essentially, that $f : A \mapsto H(X_A)$ is submodular.

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**Proof.**

Let $X$ be a random variable drawn uniformly from $S \subseteq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

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- By Shearers' Lemma,

$$2H(X) \leq H(X_{\{1,2\}}) + H(X_{\{1,3\}}) + H(X_{\{2,3\}})$$

So $2 \log_2 |S| \leq \log_2 |S_{xy}| + \log_2 |S_{xz}| + \log_2 |S_{yz}|$ □

# A matroid counting lemma

We write $[n] := \{1, \dots, n\}$ and

$$\mathbb{M}_{n,r} := \{M \text{ matroid } \mid E(M) = [n], r(M) = r\}.$$

Let $\mathcal{M}$ be a class of matroids $M$ with $E(M) \subseteq \mathbb{N}$, closed under

- contraction
- order-preserving isomorphism

We write $m_{n,r} := |\mathcal{M} \cap \mathbb{M}_{n,r}|$

**Lemma (Bansal, P., van der Pol, 2012)**

*For any $1 \leq t \leq r \leq n$:*

$$\log_2(m_{n,r} + 1) / \binom{n}{r} \leq \log_2(m_{n-t,r-t} + 1) / \binom{n-t}{r-t}$$

## Lemma (Bansal, P., van der Pol, 2012)

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## Proof.

Write $E := [n]$, and let $X$ be drawn uniformly from

$$\{\chi^{\mathcal{B}} \in \{0,1\}^{\binom{E}{r}} \mid \mathcal{B} \text{ satisfies basis exchange axiom}\}$$

- $H(X) = \log_2(m_{n,r}+1)$

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- Let $T \in \binom{E}{t}$, put

$$X/T := X_{\{B \in \binom{E}{r} | T \subseteq B\}}$$

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  Then $H(X/T) \leq \log_2(m_{n-t,r-t}+1)$
- By Shearers' Lemma, $\binom{r}{t} H(X) \leq \sum_{T \in \binom{E}{t}} H(X/T)$

□

# A bound on the number of matroids

**Lemma (Bansal, P., van der Pol, 2012)**

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In case $\mathcal{M} =$ all matroids, we have $m_{n,2}+1 \leq (n+1)^n$, hence

**Theorem (Bansal, P., van der Pol, 2012)**

$$\log m_n \leq O(\frac{\log(n)}{n}\binom{n}{\lfloor n/2 \rfloor})$$

# Part II: Counting matroids by Compression

# Matroid covers

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**Definition**

A set $\mathcal{Z} \subseteq 2^E$ *covers* $M$ if

$$\mathcal{B} = \{X \in \binom{E}{r} \mid \text{ no element of } \mathcal{Z} \text{ covers } X\}$$

If $\mathcal{Z}$ covers $M$, then $\{(F, r_M(F) \mid F \in \mathcal{Z}\}$ characterizes $M$.

A cover $\mathcal{Z}$ is a compressed description of $M$.

# Cover complexity

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**Lemma**

- $\kappa(M^*) = \kappa(M)$
- *if $M$ is a minor of $N$, then $\kappa(M) \leq \kappa(N)$*
- $\kappa(M) \leq \kappa(M \setminus e) + \kappa(M/e)$, *unless $e$ is a loop or a coloop*
- *if $M$ arises from $N$ by relaxing a circuit-hyperplane, then*

$$\kappa(M) = \kappa(N) - 1$$

# Cover complexity and counting

## Lemma

*The number of matroids* $M \in \mathbb{M}_{n,r}$ *with* $\kappa(M) \leq k$ *is at most*

$$\sum_{i=0}^{k} \binom{2^n r}{i}$$

Proof: if $\mathcal{Z}$ covers $M$, then $\{(F, r_M(F)) : F \in \mathcal{Z}\} \subseteq 2^{[n]} \times \{0, \ldots, r-1\}$ characterizes $M$.

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**Theorem**

*Suppose $\mathcal{M}$ is a class of matroids so that*

$$\max\{\kappa(M) : M \in \mathcal{M} \cap \mathbb{M}_n\} \leq O(\log(n) 2^n / n^\alpha) \text{ as } n \to \infty$$

*for some constant $\alpha > 0$. Then*

$$\log |\mathcal{M} \cap \mathbb{M}_n| \leq O(\log(n)^2 2^n / n^\alpha) \text{ as } n \to \infty.$$

# Fractional cover complexity

The *fractional cover complexity* of a matroid $M = (E, \mathcal{B})$ is $\kappa^*(M) :=$

$$\min\{\sum_F z_F \mid z : 2^E \to \mathbb{R}_+, \sum_{F:F \text{ covers } X} z_F \geq 1 \text{ for each non-basis } X \text{ of } M\}$$

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**Lemma**

*If $M \in \mathbb{M}_{n,r}$, then $\kappa(M) \leq \kappa^*(M)(\ln(\binom{n}{r}/\kappa^*(M)) + 1)$.*

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**Theorem (P., van der Pol, 2012)**

*If $\mathcal{M}$ is closed under contraction and isomorphism, then*

$$\frac{\max\{\kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n,r}\}}{\binom{n}{r}} \leq \frac{\max\{\kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n-t,r-t}\}}{\binom{n-t}{r-t}}$$

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Hence:

$$\frac{\max\{\kappa^*(M) : M \in \mathbb{M}_{n,r}\}}{\binom{n}{r}} \leq \frac{\max\{\kappa(M) : M \in \mathbb{M}_{n-r+1,1}\}}{\binom{n-r+1}{1}} \leq \frac{1}{n-r+1}$$

## Theorem

$\log m_n \leq O(\log(n)^2 2^n / n^{3/2})$.

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### Lemma

*If $M \in \mathbb{M}_{n,2}$ has no $U_{2,k}$-minor, then $\kappa(M) \leq k$.*

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Consider $\mathcal{M} =$ matroids without $N$-minor, where $N = U_{3,6}, P_6, Q_6$, or $R_6$.

## Lemma

*If $M \in \mathbb{M}_{n,3}$ has no $N$-minor, then $\kappa(M) \leq O(n)$.*

Hence:

$$\frac{\max\{\kappa^*(M) : M \in \mathbb{M}_{n,r}\}}{\binom{n}{r}} \leq \frac{\max\{\kappa(M) : M \in \mathbb{M}_{n-r+3,3}\}}{\binom{n-r+3}{3}} \leq \frac{cn}{\binom{n-r+3}{3}}$$

## Theorem

$\log|\mathcal{M} \cap \mathbb{M}_n| \leq O(\log(n)^2 2^n / n^{5/2})$.

# Even better compression of matroids..

We may construct a description of $M = (E, \mathcal{B})$ of rank $r$ consisting of

- a 'very small' set $S \subseteq \binom{E}{r}$, which determines a 'small' set $A \subseteq \binom{E}{r}$
- a 'very small' cover of all nonbases in $\binom{E}{r} \setminus A$
- a description of the bases inside $A$, using only $|A|$ bits

Here 'very small' is $O(\log(n)/n^2\binom{n}{r})$, 'small' is $\leq 2/n\binom{n}{r}$.

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