Counting Matroids, Entropy and Compression

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Motivation

Question

How many bits suffice to store any matroid of rank $r$ on $n$ elements?

By a theorem of Knuth, there are at least $2 \frac{n^r}{n}$ such matroids. So at least $\frac{n^r}{n}$ bits.

In fact, number of bits = $\log_2(\# \text{of matroids})$.
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By a theorem of Knuth, there are at least $2^{(n \choose r)/n}$ such matroids.
So at least $(n \choose r)/n$ bits. In fact, number of bits $= \log_2(\# \text{ of matroids})$

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How many bits suffice to store any matroid of rank $r$ on $n$ elements in a way that supports a rank oracle that takes $O(n^k)$ time?
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$\binom{n}{r}$ bits suffice to store bases; rank oracle takes $O(n^3)$ time from that data.
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\( \binom{n}{r} \) bits suffice to store bases; rank oracle takes \( O(n^3) \) time from that data.

Question

How do you construct a random matroid?
Motivation

'Asymptotically almost all matroids have property $\mathcal{P}$' if

$$\lim_{n \to \infty} \frac{\#\{M \in \mathcal{M}_n : M \text{ has property } \mathcal{P}\}}{\#\mathcal{M}_n} = 1$$

where $\mathcal{M}_n := \{ M \text{ matroid : } E(M) = \{1, \ldots, n\}\}$. 

The following are due to Mayhew, Newman, Welsh & Whittle (2010):

**Conjecture**
Asymptotically, almost all matroids are sparse paving.

**Conjecture**
For any $k$, asymptotically all matroids are $k$-connected.

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If $N$ is a sparse paving matroid, then a.a. matroids have an $N$-minor.
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The number of single-element extensions of any matroid of rank $r$ on $n$ elements is at most the number of single-element extensions of $U(r, n)$.
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The number of single-element extensions of any matroid of rank \( r \) on \( n \) elements is at most the number of single-element extensions of \( U(r, n) \).

Question

What determines the number of extensions of any given matroid?
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The number of single-element extensions of any matroid of rank $r$ on $n$ elements is at most the number of single-element extensions of $U(r, n)$.

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What determines the number of extensions of any given matroid?

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What determines the number of extensions of any given matroid within a minor-closed class?
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Part I: Counting matroids by Entropy
The Loomis-Whitney inequality

Theorem (Loomis & Whitney, 1949)

Let $S \subseteq \mathbb{Z}^3$ be a finite set. Then

$$|S|^2 \leq |S_{xy}| \ |S_{xz}| \ |S_{yz}|$$

where $S_{uv}$ denotes the projection of $S$ on the $uv$-plane.
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A modern proof uses *Shearers’ Entropy Lemma*
Shearers’ Entropy Lemma

A random variable taking values in a finite set $\mathcal{X}$. 

Definition

The entropy of $X$ is $H(X) := \sum_{x \in \mathcal{X}} P(X = x) \log_2 \frac{1}{P(X = x)}$.

Note: $H(X) \leq \log_2 |\mathcal{X}|$, with equality iff $X$ is drawn uniformly from $\mathcal{X}$.

If $X = (X_1, \ldots, X_n)$, we write $X_{\{a_1, \ldots, a_k\}} := (X_{a_1}, \ldots, X_{a_k})$.

Theorem (Shearer, 1986)

Suppose $X$ takes values in $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Let $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ such that $\#\{i \mid j \in A_i\} \geq k$ for $j = 1, \ldots, n$. Then $kH(X) \leq m \sum_{i=1}^n H(X_{A_i})$.

Proof: essentially, that $f : A \mapsto H(X_{A_i})$ is submodular.
Shearers’ Entropy Lemma

$X$ a random variable taking values in a finite set $\mathcal{X}$.

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**Proof.**

Let $X$ *be a random variable drawn uniformly from* $S \subseteq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.  

- $H(X) = \log_2 |S|$
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- $H(X) = \log_2 |S|$
- $H(X_{\{1,2\}}) \leq \log_2 |S_{xy}|$, $H(X_{\{1,3\}}) \leq \log_2 |S_{xz}|$, $H(X_{\{2,3\}}) \leq \log_2 |S_{yz}|$
The proof of Loomis-Whitney

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- $H(X_{\{1,2\}}) \leq \log_2 |S_{xy}|$, $H(X_{\{1,3\}}) \leq \log_2 |S_{xz}|$, $H(X_{\{2,3\}}) \leq \log_2 |S_{yz}|$
- By Shearers’ Lemma,

$$2H(X) \leq H(X_{\{1,2\}}) + H(X_{\{1,3\}}) + H(X_{\{2,3\}})$$

So $2 \log_2 |S| \leq \log_2 |S_{xy}| + \log_2 |S_{xz}| + \log_2 |S_{yz}|$
A matroid counting lemma

We write $[n] := \{1, \ldots, n\}$ and

$$\mathcal{M}_{n,r} := \{M \text{ matroid} \mid E(M) = [n], r(M) = r\}.$$

Let $\mathcal{M}$ be a class of matroids $M$ with $E(M) \subseteq \mathbb{N}$, closed under

- contraction
- order-preserving isomorphism

We write $m_{n,r} := |\mathcal{M} \cap \mathcal{M}_{n,r}|$

Lemma (Bansal, P., van der Pol, 2012)

For any $1 \leq t \leq r \leq n$:

$$\log_2(m_{n,r} + 1)/\binom{n}{r} \leq \log_2(m_{n-t,r-t} + 1)/\binom{n-t}{r-t}$$
Lemma (Bansal, P., van der Pol, 2012)

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Lemma (Bansal, P., van der Pol, 2012)

For any $1 \leq t \leq r \leq n$: $\log_2(m_{n,r} + 1)/C_n^r \leq \log_2(m_{n-t,r-t} + 1)/C_{n-t}^{r-t}$

Proof.

Write $E := [n]$, and let $X$ be drawn uniformly from

$$\{\chi^B \in \{0, 1\}^\binom{E}{r} \mid B \text{ satisfies basis exchange axiom}\}$$

- $H(X) = \log_2(m_{n,r} + 1)$
Lemma (Bansal, P., van der Pol, 2012)

For any \(1 \leq t \leq r \leq n\): \(\log_2(m_{n,r} + 1) / \binom{n}{r} \leq \log_2(m_{n-t, r-t} + 1) / \binom{n-t}{r-t}\)

Proof.

Write \(E := [n]\), and let \(X\) be drawn uniformly from

\[
\{ \chi^B \in \{0, 1\}^{E(r)} | B \text{ satisfies basis exchange axiom} \}
\]

- \(H(X) = \log_2(m_{n,r} + 1)\)
- Let \(T \in \binom{E}{t}\), put

\[
X / T := X_{\{B \in \binom{E}{r} | T \subseteq B\}}
\]

Then \(H(X / T) \leq \log_2(m_{n-t, r-t} + 1)\)
Lemma (Bansal, P., van der Pol, 2012)

For any $1 \leq t \leq r \leq n$: \[
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\]

Proof.

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  Then $H(X/T) \leq \log_2 (m_{n-t,r-t} + 1)$
- By Shearers’ Lemma, \[
  \binom{r}{t} H(X) \leq \sum_{T \in \binom{E}{t}} H(X/T)
  \]
A bound on the number of matroids

Lemma (Bansal, P., van der Pol, 2012)

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$$\log_2(m_{n,r} + 1)/\binom{n}{r} \leq \log_2(m_{n-t,r-t} + 1)/\binom{n-t}{r-t}$$

In case $M =$ all matroids, we have $m_{n,2} + 1 \leq (n + 1)^n$, hence

Theorem (Bansal, P., van der Pol, 2012)

$$\log m_n \leq O\left(\frac{\log(n)}{n} \left(\binom{n}{\lfloor n/2 \rfloor}\right)\right)$$
Part II: Counting matroids by Compression
Matroid covers

Let $M = (E, B)$ be a matroid of rank $r$. 
Matroid covers

Let $M = (E, \mathcal{B})$ be a matroid of rank $r$.

**Definition**

A dependent set $X$ is *covered* by a set $F$ if $|X \cap F| > r_M(F)$.

Then $F$ certifies that $X$ is dependent.
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**Definition**

A set $Z \subseteq 2^E$ covers $M$ if

$$B = \{ X \in \binom{E}{r} \mid \text{no element of } Z \text{ covers } X \}$$

If $Z$ covers $M$, then $\{(F, r_M(F)) \mid F \in Z\}$ characterizes $M$.

A cover $Z$ is a compressed description of $M$. 
Cover complexity

Definition (Cover complexity)

The *cover complexity* of $M$ is $\kappa(M) := \min\{|Z| : Z \text{ covers } M\}$. 

Lemma

If $M$ is a minor of $N$, then $\kappa(M) \leq \kappa(N)$.

If $M$ arises from $N$ by relaxing a circuit-hyperplane, then $\kappa(M) = \kappa(N) - 1$. 

Unless $e$ is a loop or a coloop.
Cover complexity

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The *cover complexity* of $M$ is $\kappa(M) := \min\{|Z| : Z \text{ covers } M\}$.

**Lemma**

- $\kappa(M^*) = \kappa(M)$
- If $M$ is a minor of $N$, then $\kappa(M) \leq \kappa(N)$
- $\kappa(M) \leq \kappa(M \setminus e) + \kappa(M/e)$, unless $e$ is a loop or a coloop
- If $M$ arises from $N$ by relaxing a circuit-hyperplane, then
  \[ \kappa(M) = \kappa(N) - 1 \]
Lemma

The number of matroids $M \in \mathcal{M}_{n,r}$ with $\kappa(M) \leq k$ is at most

$$\sum_{i=0}^{k} \binom{2^nr}{i}$$

Proof: if $\mathcal{Z}$ covers $M$, then $\{(F, r_M(F)) : F \in \mathcal{Z}\} \subseteq 2^n \times \{0, \ldots, r - 1\}$ characterizes $M$. 
Cover complexity and counting

Lemma

The number of matroids \( M \in \mathbb{M}_{n,r} \) with \( \kappa(M) \leq k \) is at most

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\sum_{i=0}^{k} \binom{2^n r}{i}
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Proof: if \( \mathcal{Z} \) covers \( M \), then \( \{(F, r_M(F)) : F \in \mathcal{Z}\} \subseteq 2^n \times \{0, \ldots, r - 1\} \) characterizes \( M \).

Theorem

Suppose \( \mathcal{M} \) is a class of matroids so that

\[
\max\{\kappa(M) : M \in \mathcal{M} \cap \mathbb{M}_{n}\} \leq O(\log(n)2^n/n^\alpha) \text{ as } n \to \infty
\]

for some constant \( \alpha > 0 \). Then

\[
\log |\mathcal{M} \cap \mathbb{M}_n| \leq O(\log(n)^2 2^n/n^\alpha) \text{ as } n \to \infty.
\]
Fractional cover complexity

The fractional cover complexity of a matroid $M = (E, \mathcal{B})$ is $\kappa^*(M) :=$

$$\min \{ \sum_{F} z_F \mid z : 2^E \to \mathbb{R}_+, \sum_{F: F \text{ covers } X} z_F \geq 1 \text{ for each non-basis } X \text{ of } M \}$$

This the LP relaxation of the 'IP' that defines $\kappa(M)$. So: $\kappa^*(M) \leq \kappa(M)$. 
Fractional cover complexity

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Lemma

If $M \in \mathbb{M}_{n,r}$, then $\kappa(M) \leq \kappa^*(M) (\ln(\binom{n}{r}/\kappa^*(M)) + 1)$.
Fractional cover complexity

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Lemma

If $M \in \mathcal{M}_{n,r}$, then $\kappa(M) \leq \kappa^*(M)(\ln((n/r)/\kappa^*(M)) + 1)$.

Theorem (P., van der Pol, 2012)

If $\mathcal{M}$ is closed under contraction and isomorphism, then

$$\frac{\max\{\kappa^*(M) : M \in \mathcal{M} \cap \mathcal{M}_{n,r}\}}{\binom{n}{r}} \leq \frac{\max\{\kappa^*(M) : M \in \mathcal{M} \cap \mathcal{M}_{n-t,r-t}\}}{\binom{n-t}{r-t}}$$
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Consider \( \mathcal{M} = \) all matroids.
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Consider $\mathcal{M} = \text{all matroids}$.

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If $M \in \mathcal{M}_{n,1}$, then $\kappa(M) \leq 1$. 
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Consider $\mathcal{M} =$ all matroids.

Lemma

If $M \in \mathbb{M}_{n,1}$, then $\kappa(M) \leq 1$.

Hence:

$$\max\left\{ \kappa^*(M) : M \in \mathbb{M}_{n,r} \right\} \leq \max\left\{ \kappa(M) : M \in \mathbb{M}_{n-r+1,1} \right\} \leq \frac{1}{n-r+1}$$

Theorem

$$\log m_n \leq O(\log(n)^2 2^n / n^{3/2}).$$
Theorem (P., van der Pol, 2012)

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$$\max \{ \kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n,r} \} \leq \frac{\max \{ \kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n-t,r-t} \}}{\binom{n}{r}} \leq \frac{\max \{ \kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n-t,r-t} \}}{\binom{n-t}{r-t}}$$
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Consider $\mathcal{M} =$ matroids without $U_{2,k}$-minor.
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Consider \( \mathcal{M} = \) matroids without \( U_{2,k} \)-minor.

Lemma

If \( M \in \mathbb{M}_{n,2} \) has no \( U_{2,k} \)-minor, then \( \kappa(M) \leq k \).
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Consider \( \mathcal{M} = \) matroids without \( U_{2,k} \)-minor.

Lemma

If \( M \in \mathcal{M}_{n,2} \) has no \( U_{2,k} \)-minor, then \( \kappa(M) \leq k \).

Hence:

\[
\max\{ \kappa^*(M) : M \in \mathcal{M}_{n,r} \} \leq \max\{ \kappa(M) : M \in \mathcal{M}_{n-r+2,2} \} \leq \frac{k}{\binom{n-r+2}{2}}
\]

Theorem

\[
\log |\mathcal{M} \cap \mathcal{M}_n| \leq O(\log(n)^22^n/n^{5/2}).
\]
Theorem (P., van der Pol, 2012)

If \( \mathcal{M} \) is closed under contraction and isomorphism, then

\[
\frac{\max\{ \kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n,r} \}}{\binom{n}{r}} \leq \frac{\max\{ \kappa^*(M) : M \in \mathcal{M} \cap \mathbb{M}_{n-t,r-t} \}}{\binom{n-t}{r-t}}
\]

Consider \( \mathcal{M} = \) matroids without \( N \)-minor, where \( N = U_{3,6}, P_6, Q_6, \) or \( R_6 \).

Lemma

If \( M \in \mathbb{M}_{n,3} \) has no \( N \)-minor, then \( \kappa(M) \leq O(n) \).

Hence:

\[
\frac{\max\{ \kappa^*(M) : M \in \mathbb{M}_{n,r} \}}{\binom{n}{r}} \leq \frac{\max\{ \kappa(M) : M \in \mathbb{M}_{n-r+3,3} \}}{\binom{n-r+3}{3}} \leq \frac{cn}{\binom{n-r+3}{3}}
\]

Theorem

\[
\log |\mathcal{M} \cap \mathbb{M}_n| \leq O(\log(n)^2 2^n / n^{5/2}).
\]
Even better compression of matroids.

We may construct a description of $M = (E, B)$ of rank $r$ consisting of

- a 'very small' set $S \subseteq \binom{E}{r}$, which determines a 'small' set $A \subseteq \binom{E}{r}$
- a 'very small' cover of all nonbases in $\binom{E}{r} \setminus A$
- a description of the bases inside $A$, using only $|A|$ bits

Here 'very small' is $O(\log(n)/n^2 \binom{n}{r})$, 'small' is $\leq 2/n\binom{n}{r}$. 

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Such compression yields:

**Theorem (Bansal, P., van der Pol, 2012)**

$$\log m_n \leq \frac{2}{n} \left(\binom{n}{\lfloor n/2 \rfloor}\right) (1 + o(1))$$
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**Theorem (Bansal, P., van der Pol, 2012)**

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**Theorem (Knuth, 1974)**

$$\log m_n \geq \frac{1}{n} \left( \frac{n}{\lfloor n/2 \rfloor} \right)$$