Inequivalent representations over GF(7)

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Equivalence of representations

Let $M = (E, \mathcal{I})$ be a matroid.

A and A' are matrices over a field that represent M: the columns of A and A' are labeled by the elements of E and $X \subseteq E$ is in $\mathcal{I}$ if and only if X labels a linearly independent set of columns.

A and A' are equivalent if one is obtained from the other by:

- adding a row to another,
- scaling rows/columns by non-zero numbers,
- permuting rows and columns,
- deleting/adding zero rows.

Let $n_q(M)$ be the number of equivalence classes of matrices that represent M over $\mathrm{GF}(q)$.

Kahn's conjecture

If $q = 2, 3, 4$, then $n_q(M) \leq 2$, for any 3-connected GF(q)-representable matroid M.

Conjecture (Kahn – 1988)

Let q be a prime power. There exists integer N_q such that

$$n_q(M) \leq N_q$$

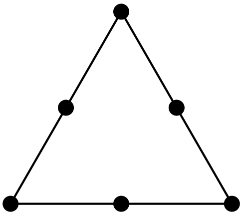
for any 3-connected GF(q)-representable matroid M.

Fixed elements

Let e, e' be elements in matroid M. If the transposition of e and e' is an automorphism of M, e and e' are clones.

If e is an element of M, and M' is single-element extension of M by e' such that e and e' are clones, then M' is a clonal extension.

If such an M' exists with $\{e, e'\}$ independent, then e is free, otherwise e is fixed.



Fixed elements

Assume e is fixed in M, and both

$$\begin{bmatrix} A & \begin{array}{|c} e \\ \mathbf{x} \end{array} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & \begin{array}{|c} e \\ \mathbf{x}' \end{array} \end{bmatrix}$$

represent M. Then

$$\mathbf{x}' = \lambda \mathbf{x}$$

for some non-zero λ.

So in this case,

$$n_q(M) \leq n_q(M \backslash e).$$

If e is cofixed (fixed in M^*), then $n_q(M) \leq n_q(M/e)$.

Skeletons

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Assume we want to bound

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If M' is Blah-connected, and is produced from M by a sequence of:

- deleting a fixed element, where the deletion is Blah-connected,
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M' is a Blah-skeleton if no further moves of this type can be performed.

Skeletons

M' is a Blah-skeleton if

- M' is Blah-connected,
- if e is fixed in M', then $M' \backslash e$ is not Blah-connected,
- if e is cofixed in M', then M'/e is not Bah-connected.

$$\max\{n_q(M) \mid M \text{ is Blah-connected, GF}(q)\text{-representable}\} \leq \max\{n_q(M') \mid M' \text{ is a GF}(q)\text{-representable Blah-skeleton}\}$$

Therefore the aim is to characterise GF(q)-representable Blah-skeletons. (We hope there are finitely many of them.)

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This corresponds to a negative answer to Kahn's conjecture.

Theorem (Oxley, Vertigan, Whittle – 1996)

If $q = 2, 3, 4, 5$, then $n_q(M) \leq 6$ for all 3-connected GF(q)-representable matroids M.

If q is a prime power and $q \geq 7$, then

$$\{n_q(M) \mid M \text{ is 3-connected and GF}(q)\text{-representable}\}$$

contains arbitrarily large integers.

Skeletons

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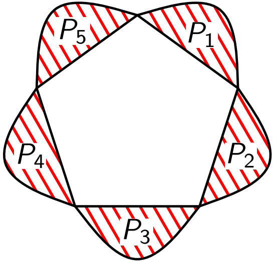
5-coherence = no swirl-like 5-flower



$$\lambda(P_i) = 2$$

$$\lambda(P_i \cup P_j) = \begin{cases} 2 \text{ if } P_i \text{ and } P_j \text{ are consecutive} \\ 3 \text{ otherwise} \end{cases}$$

Skeletons

If Blah-connectivity $=$ 5-coherent, then there is a chain theorem that enables us to find all skeletons inductively.

Theorem (Geelen, Whittle)

Let M be a non-empty (5-coherent) skeleton. Then M has a minor M' such that M' is a skeleton and $|E(M)| - |E(M)'| \leq 4$.

If $|E(M)| - |E(M')| > 1$, then we have strong information about how M' is obtained from M.

Skeletons

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If p is a prime, then there is an integer N_p such that

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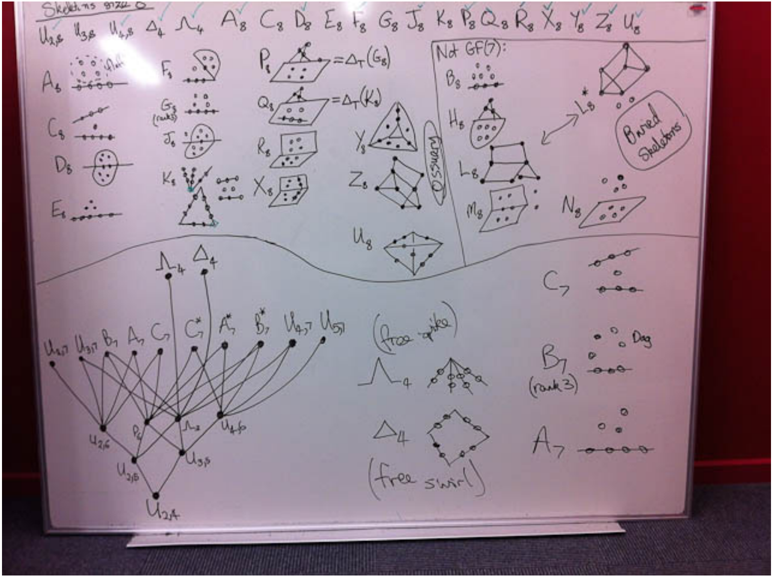
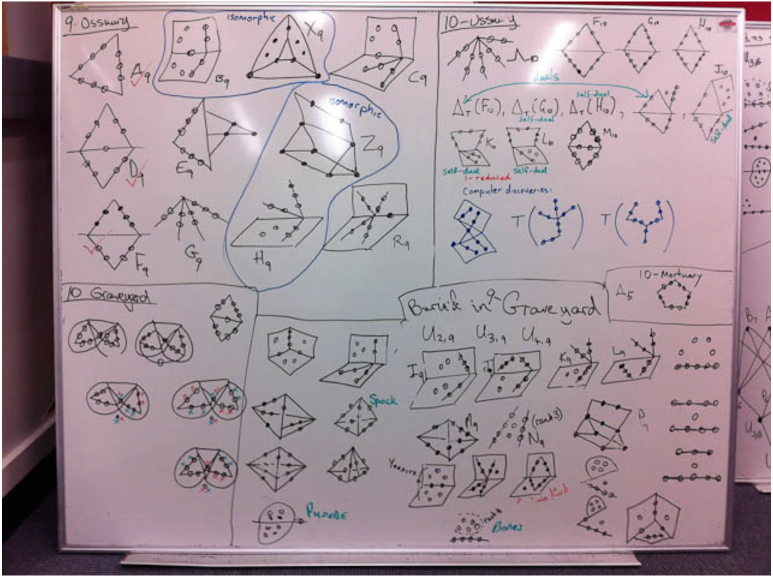
Question

What is N_7?

GF(7)-representable skeletons



GF(7)-representable skeletons



GF(7)-representable skeletons

Numbers of GF(7)-representable skeletons.

Size of ground set	Number of skeletons
4	1
5	2
6	4
7	10
8	28
9	18
10	20
11	16
12	28

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- Issues with computer search.
  - Have to find all possible representations of skeletons.
  - Search space is large, need to use structure from the Geelen/Whittle chain theorem to reduce it.