Seymour's 1-flowing Conjecture

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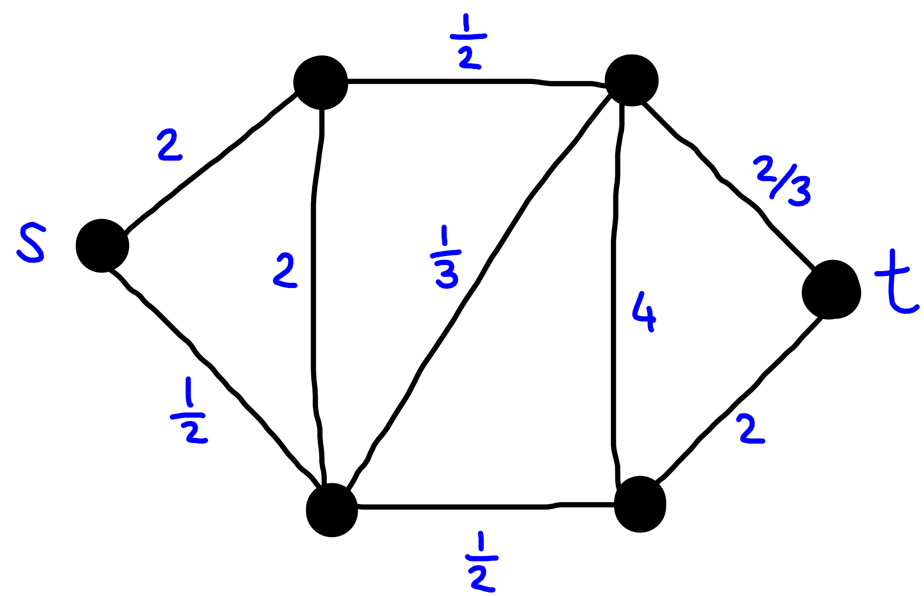
(Joint work with J. Kung, D. Mayhew and G. Royle)

Max flow-min cut



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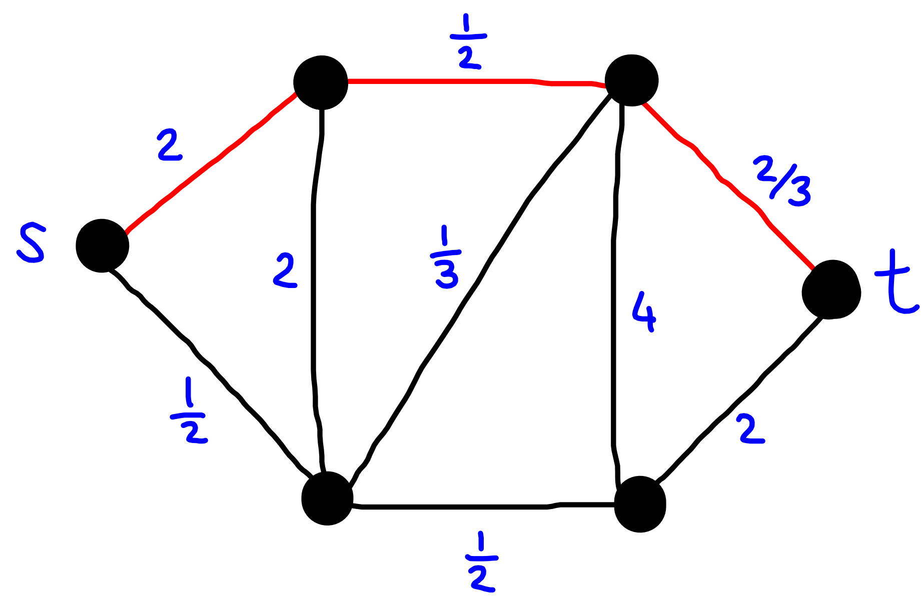
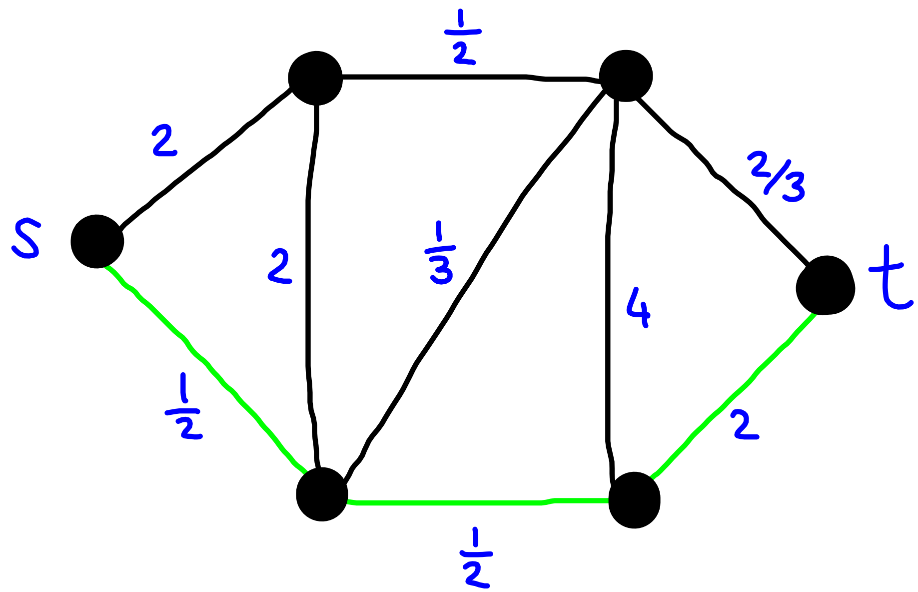


$\frac{1}{2}$

Max flow - min cut



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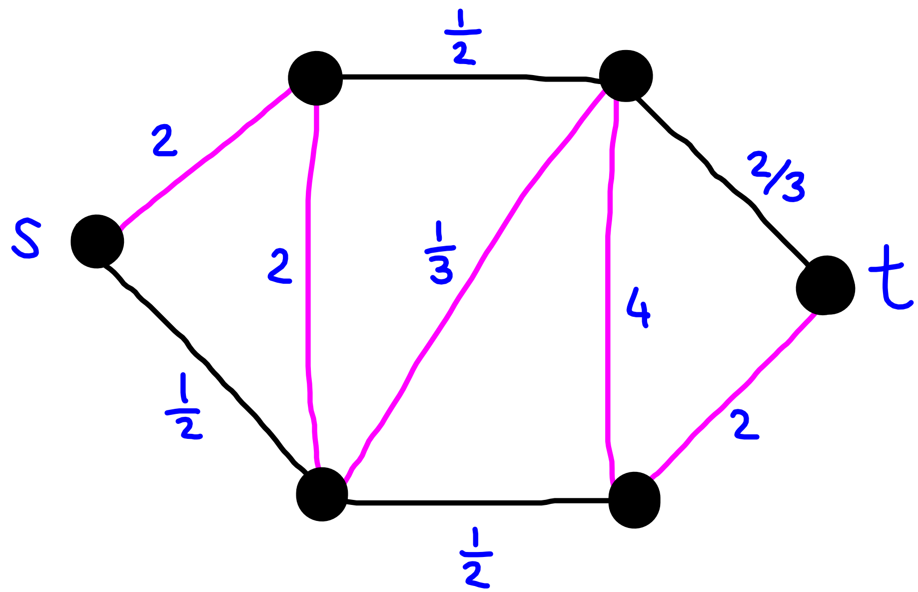
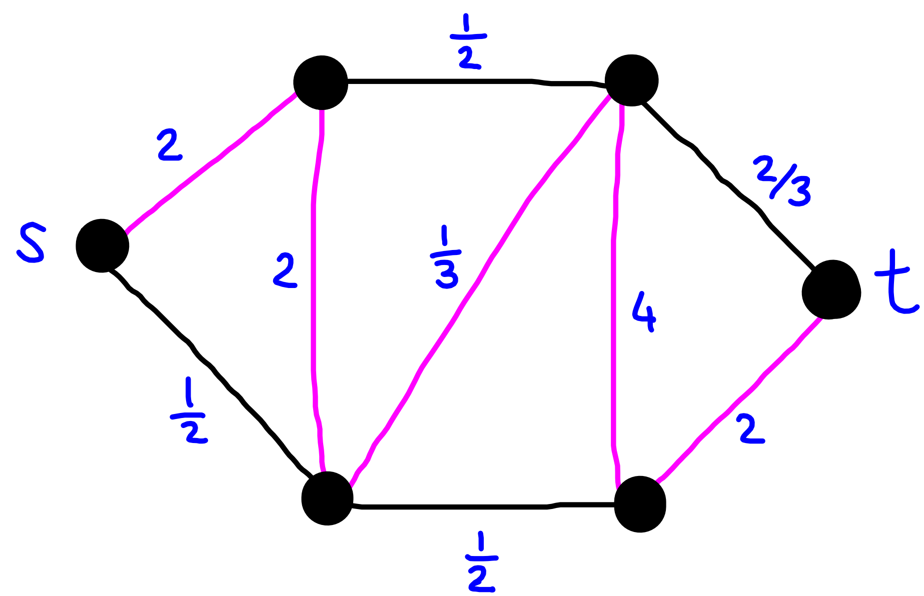
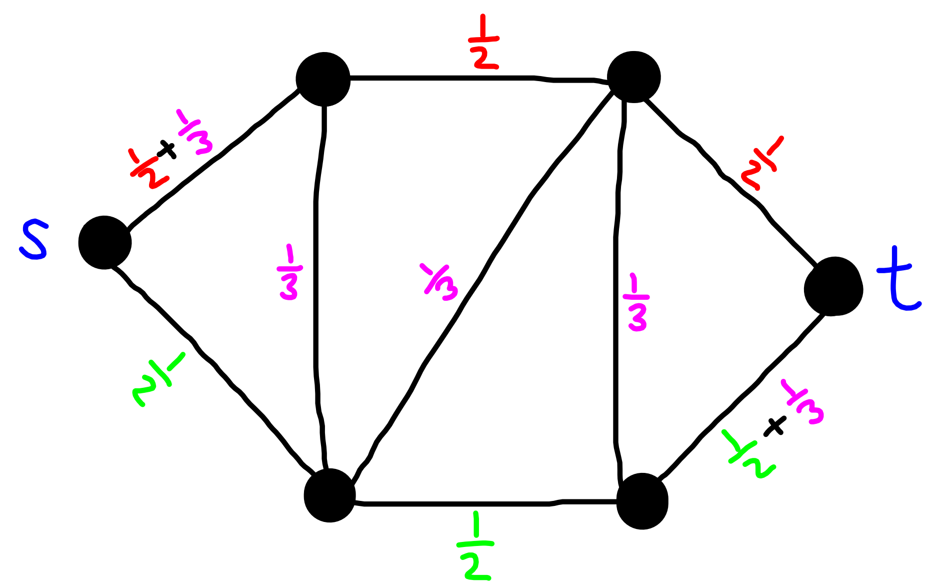


1/3

Max flow-min cut

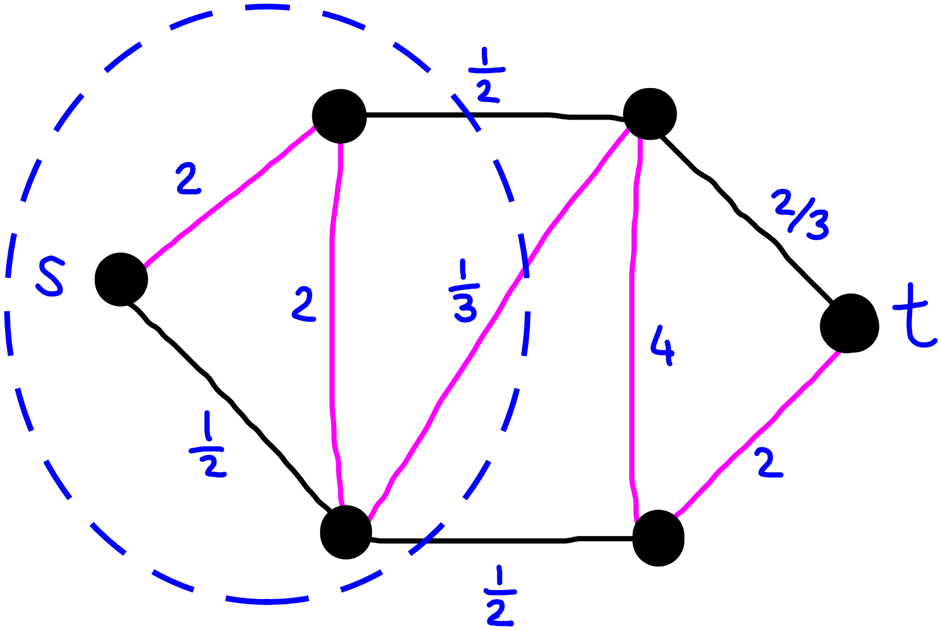




flow:

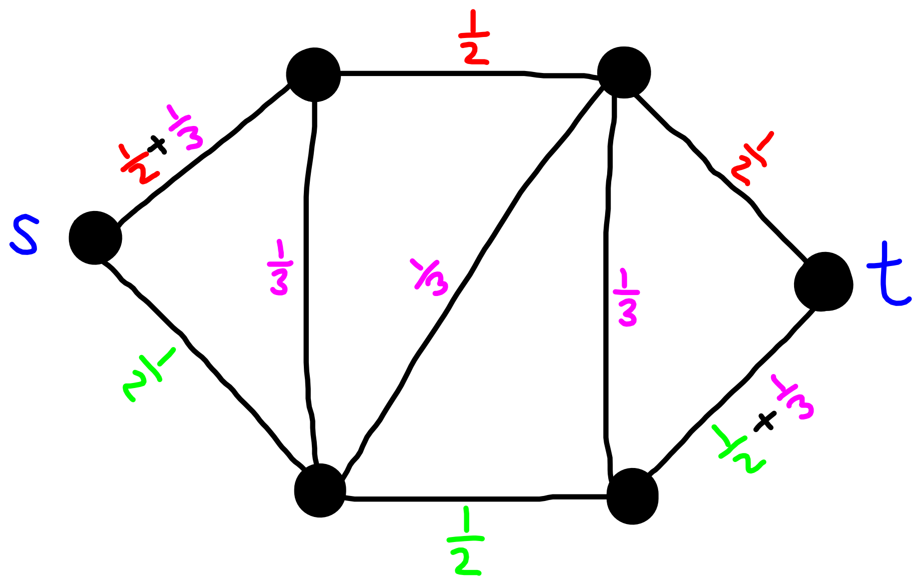
$\frac{1}{2} + \frac{1}{2} + \frac{1}{3}$

Max flow–min cut



cut:

$\frac{1}{2}+\frac{1}{3}+\frac{1}{2}$



flow:

$\frac{1}{2}+\frac{1}{2}+\frac{1}{3}$

Graph $G=(V,E)$, $s,t \in V$, $c \in \mathbb{R}_+^E$

min st-cut (IP) $\begin{cases} \min & c^T x \\ \text{s.t.} & x(P) \geq 1, \quad \text{for all st-paths } P \\ & x \in \{0,1\}^E \end{cases}$

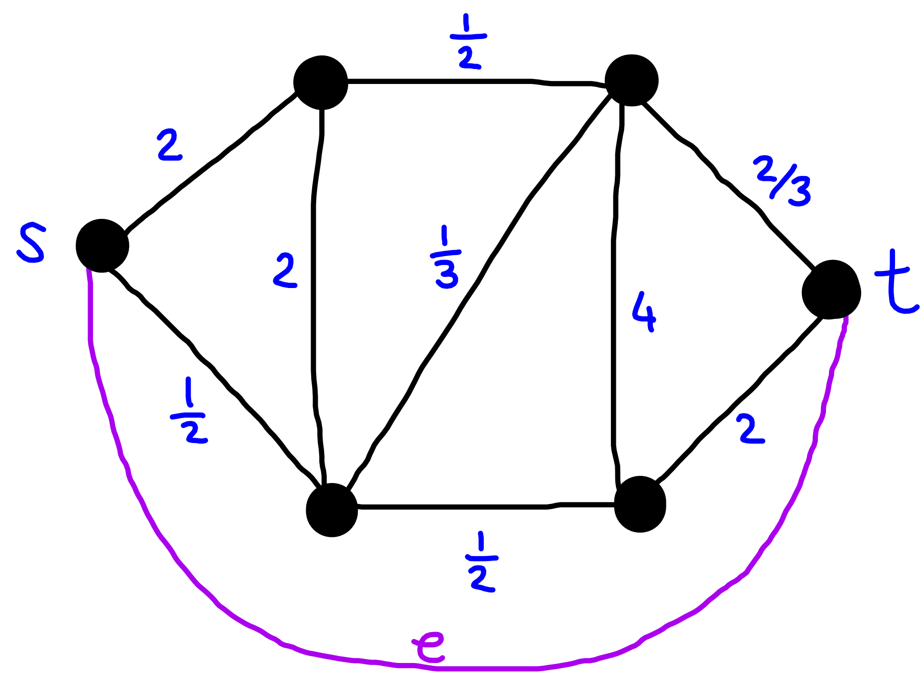
max st-flow (D) $\begin{cases} \max & \mathbb{1}^T y \\ \text{s.t.} & \sum y_P : f \in P,\ P \text{ st-path} \leq c_f \quad \text{for all } f \in E \\ & y \geq 0 \end{cases}$

Ford-Fulkerson : (IP) = (D) for all $c \in \mathbb{R}_+^E$

Max flow-min cut

G':

$M = M(G')$



P is an st-path $\iff$ $P \cup e \in \mathcal{C}(M)$

The matroid version:

$M=(E,\mathcal{C})$ a matroid, $e\in E$, $c\in\mathbb{R}_+^{E\setminus e}$.

$$(\mathrm{IP})\begin{cases}\min & c^Tx\\ \text{s.t.} & x(P)\geq 1, \qquad \text{for all } P\subseteq E\setminus e:\ P\cup e\in\mathcal{C}\\ & x\in\{0,1\}^{E\setminus e}\end{cases}$$

$$(\mathrm{D})\begin{cases}\max & \mathbb{1}^Ty\\ \text{s.t.} & \sum\{y_P: f\in P,\ P\cup e\in\mathcal{C}\}\leq c_f, \qquad \text{for all } f\in E\setminus e\\ & y\geq 0\end{cases}$$

Qu.1: when is (IP) = (D) for all c? (e-flowing matroids)

Qu.2: when is (IP) = (D) for all c's and all elements e? (1-flowing)

Seymour's 1-flowing conjecture (1977)

A binary matroid M is 1-flowing iff $M \in \mathcal{E}x(AG(3,2), T_{11}, T_{11}^*)$.

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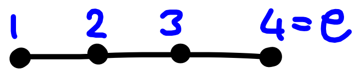
Only binary matroids because $U_{2,4}$ is not e-flowing :



$P \in \{\{1,2\}, \{2,3\}, \{1,3\}\}$, pick $c=(1,1,1)^T$

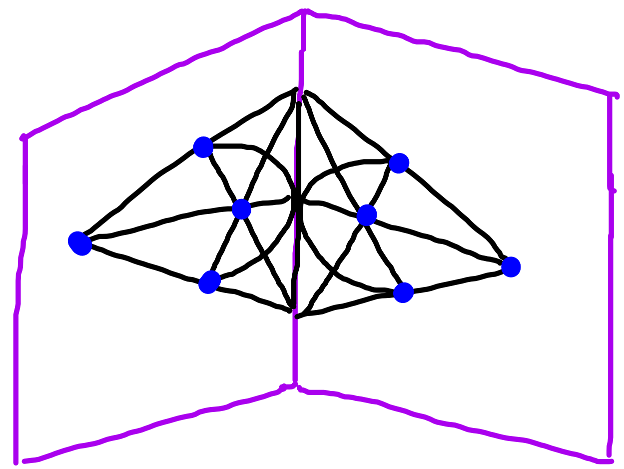
$$(IP)\quad \begin{aligned} \min\ & x_1+x_2+x_3 \\ \text{s.t. } & x_1+x_2 \geq 1 \\ & x_1+x_3 \geq 1 \\ & x_2+x_3 \geq 1 \\ & x_i \in \{0,1\} \end{aligned} \qquad\qquad (D)\quad \begin{aligned} \max\ & y_{12}+y_{23}+y_{13} \\ \text{s.t. } & y_{12}+y_{13} \leq 1 , \\ & y_{12}+y_{23} \leq 1 \\ & y_{13}+y_{23} \leq 1 \\ & y \geq 0 \end{aligned}$$

$x_1=x_2=1,\ x_3=0$

$y_{12}=y_{23}=y_{13}=\frac{1}{2}$

AG(3,2)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Self dual, no triangles, transitive aut. group,

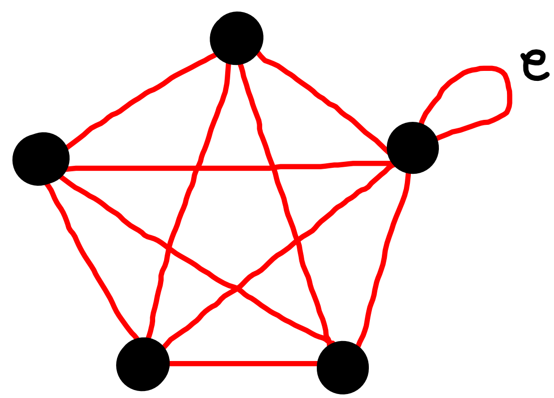
$AG(3,2)\backslash e \simeq F_7^*$, $AG(3,2)/e \simeq F_7$.

Not e-flowing for any e.

T_{11}

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

even cycle matroid



$\geq F_7^*$

Not e-flowing for this choice of e.

What do we know about 1-flowing matroids and the conjecture?

- closed under minors (Seymour)
- closed under duality (Lehman)
- holds for graphs (Ford-Fulkerson)
- holds for lifts and projections of graphic matroids (Guenin)
- any minimal counterexample must be internally 4-connected. (Cornuéjols, Guenin)

What the data suggest...

Max-size binary matroids in $Ex(AG(3,2))$ (KMPR)

Let $\mathcal{M} = \{$simple binary matroids with no $AG(3,2)$-minor$\}$.

If $M \in \mathcal{M}$ has rank $r \geq 5$, then $|M| \leq \binom{r+1}{2}$.

Moreover, if $r \geq 6$, the unique maximum sized matroid is $M(K_{r+1})$.

If $M \in \mathcal{M}$ has rank ≥ 6 and is not regular, then $|M| \leq \binom{r}{2} + 4$.

Moreover, if $r \geq 7$ the unique maximum sized non-regular matroid is the generalized parallel connection along a line of $M(K_r)$ and F_7.

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Delta-wye operation

Binary matroid M with a triangle a, b, c:

$$\begin{array}{c} \begin{array}{cccc} a & b & c & \end{array} \\ \left[\begin{array}{c|c|c|c} x & y & z & \quad A \quad \end{array}\right] \end{array} \qquad\qquad x+y+z=\underline{0}$$

$$\Big\downarrow\, \Delta\text{-}Y$$

$$\begin{array}{c} \begin{array}{cccc} a' & b' & c' & \end{array} \\ \left[\begin{array}{c|c|c|c} x & y & z & \quad A \quad \\ \hline 1 & 1 & 1 & 0 \cdots\cdots 0 \end{array}\right] \end{array}$$

Wye-delta operation

Binary matroid M with a triad a', b', c'

$$\begin{array}{cccc} a & b & c & \\ \left[\begin{array}{c|c|c|c} y+z & x+z & x+y & A \end{array}\right. & & & \left.\right] \end{array}$$

$$\uparrow \; Y\text{-}\Delta$$

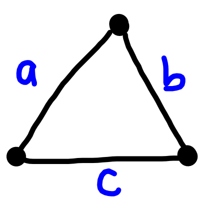
$$\begin{array}{cccc} a' & b' & c' & \\ \left[\begin{array}{c|c|c|c} x & y & z & A \end{array}\right. & & & \left.\right] \end{array}$$

a', b', c' triad

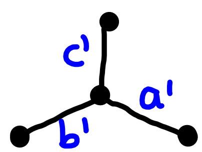
Wye-delta operation

Claim: $\mathrm{Ex}(AG(3,2))$ is closed under Δ-Y and Y-Δ.

pf: Suppose N is obtained from M by a Δ-Y, creating a triad a', b', c'. Suppose $AG(3,2) \leq N$. Since $AG(3,2)$ has no triads and is cosimple, w.m.a. that $AG(3,2) \leq N/a'$.



M



N

$N/a' \cong M \backslash a$

So $AG(3,2) \leq M$.

The claim follows by duality.

Claim: $\mathcal{E}x(AG(3,2))$ is closed under Δ-Y and Y-Δ.

Similarly, $\mathcal{E}x(AG(3,2), T_{11}, T_{11}^*)$ is closed under Δ-Y and Y-Δ.

Truemper:

A binary matroid M is almost regular if M is not regular, $E(M) = E_D \dot\cup E_C$, and

- $e \in E_D \Rightarrow M \setminus e$ is regular,
- $e \in E_C \Rightarrow M / e$ is regular,
- $E_D, E_C \neq \emptyset$,
- $|C \cap E_C|$ is even for all circuits C,
- $|D \cap E_D|$ is even for all cocircuits D.

Truemper: every almost regular matroid may be obtained from F_7 or N_{11} by a sequence of Δ-Y steps.

Δ-Y steps: Δ-Y, Y-Δ, parallel and series classes replacements.

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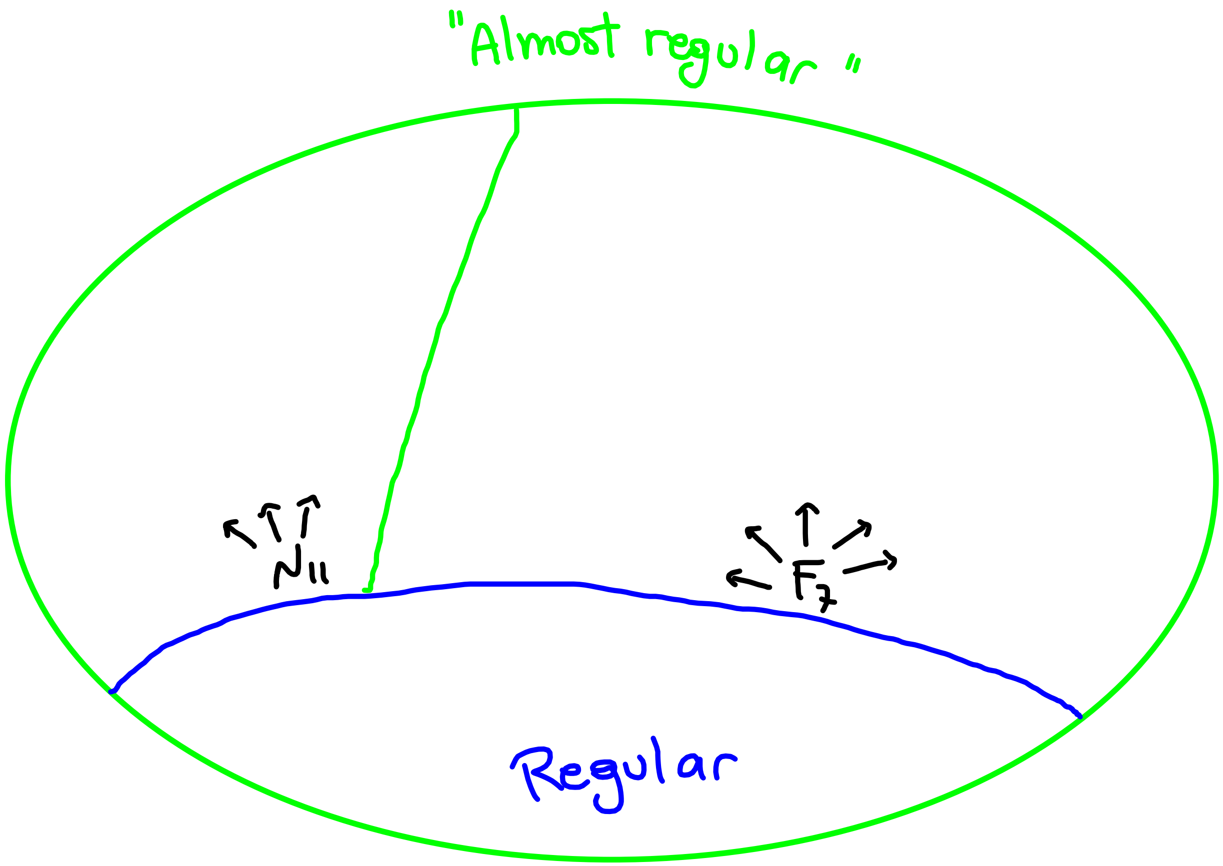
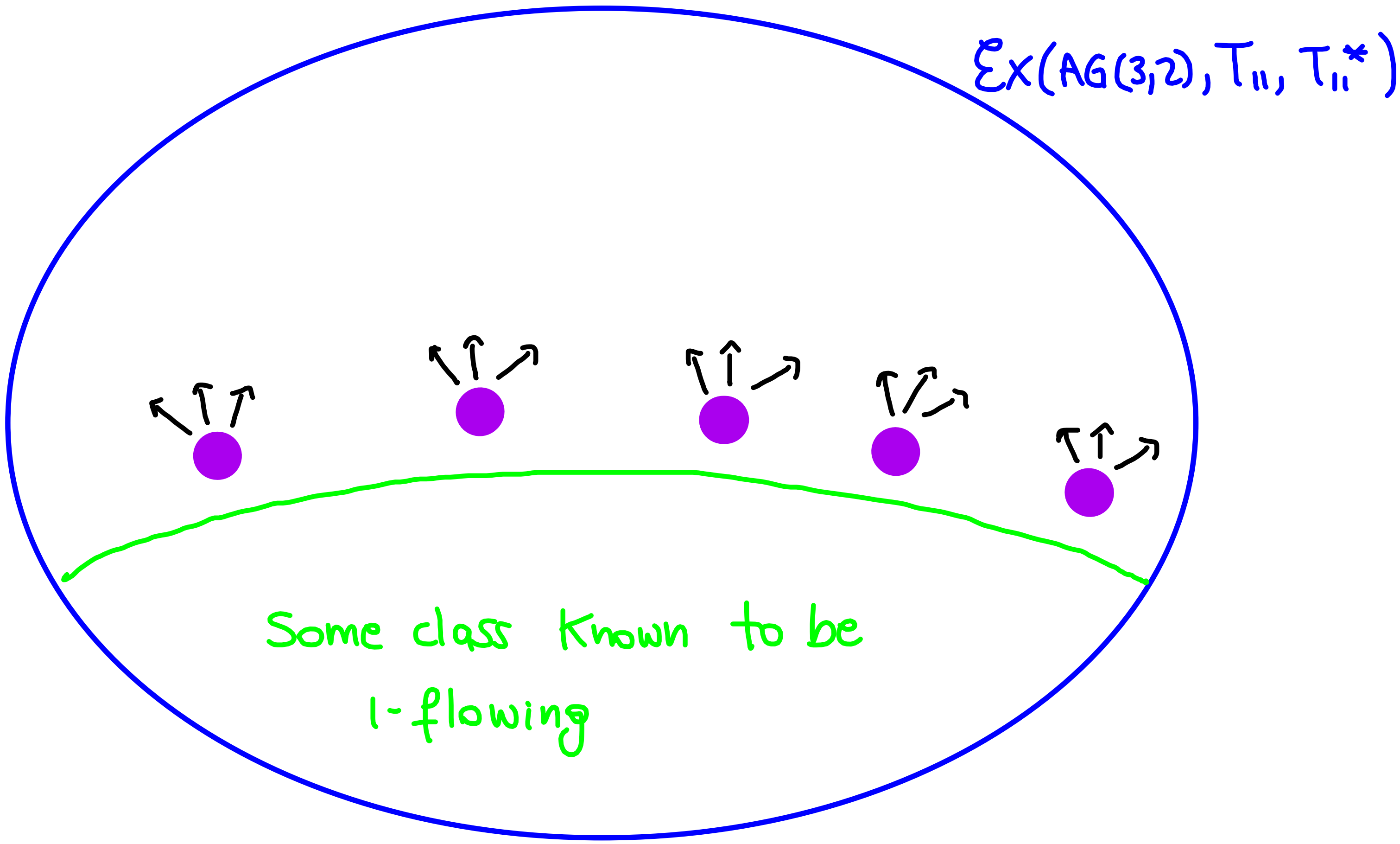


Possible strategy:

1) Prove a Truemper-type thm for $\mathrm{Ex}(AG(3,2), T_{11}, T_{11}^*)$.



Possible strategy:

1) Prove a Truemper-type thm for $\mathrm{Ex}(AG(3,2), T_{11}, T_{11}^*)$.

2) Prove that the basic matroids ● are 1-flowing.

3) Prove that the property of being 1-flowing is closed under Δ-Y steps.

Hope: see data ...

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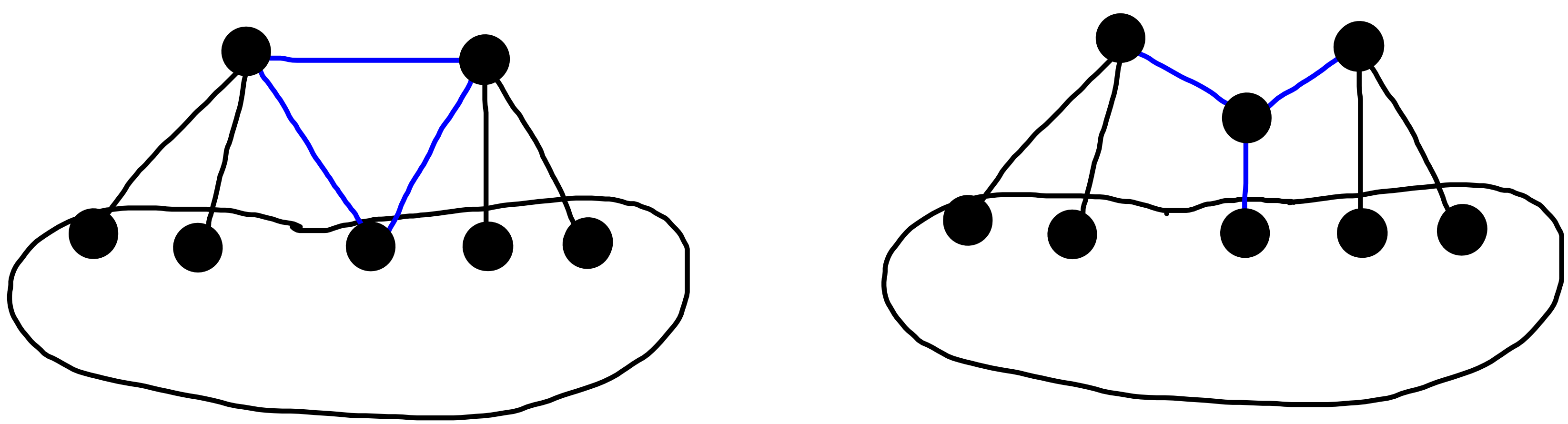
Doom: Δ-Y steps are difficult!



We need to understand Δ-Y steps.

Thank you!